Discrete-Time

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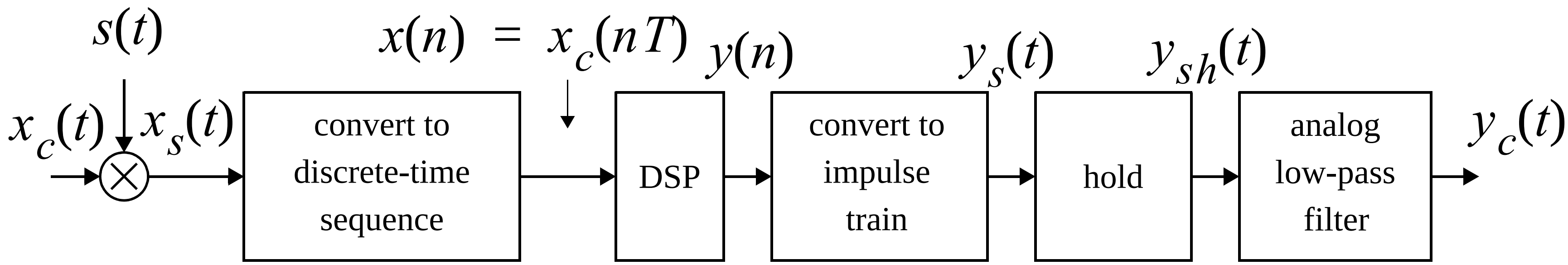
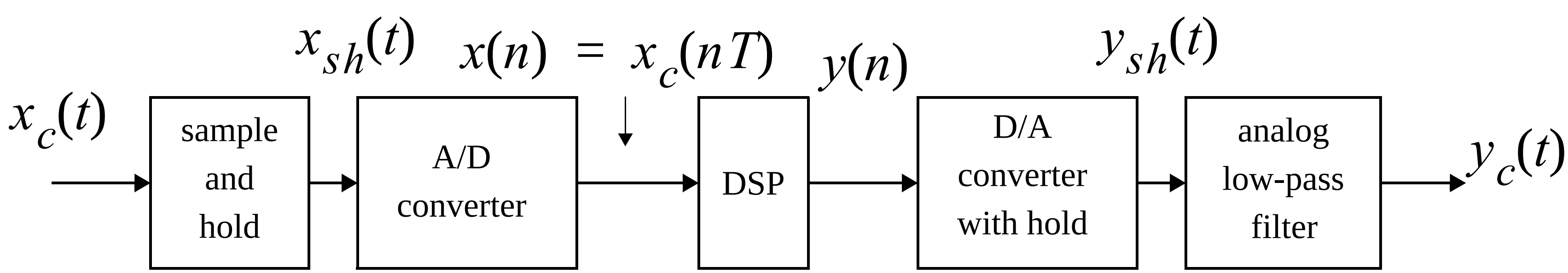
Overview of Some Signal Spectra

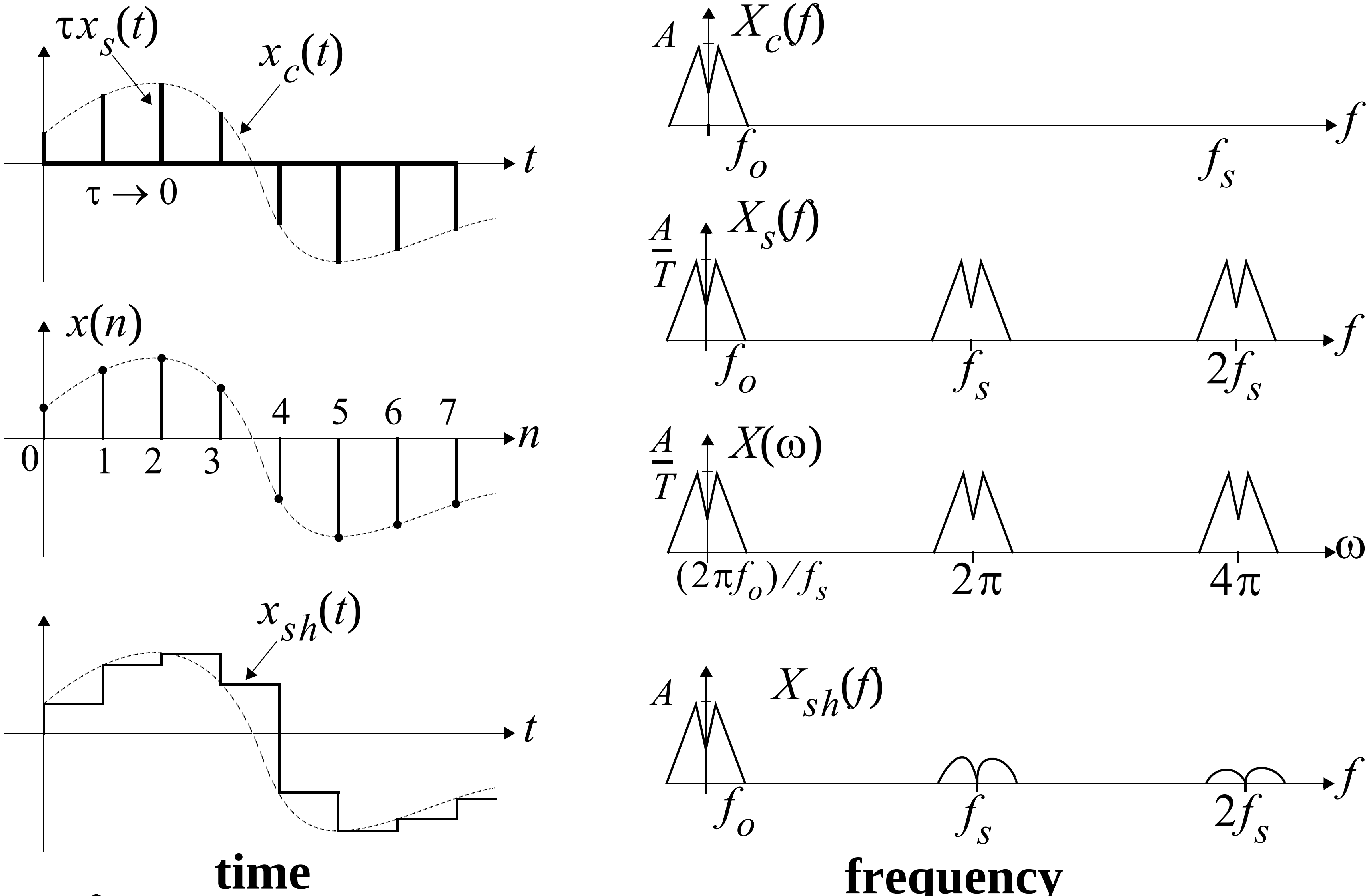


Conceptual



Typical Implementation

Example Signal Spectra



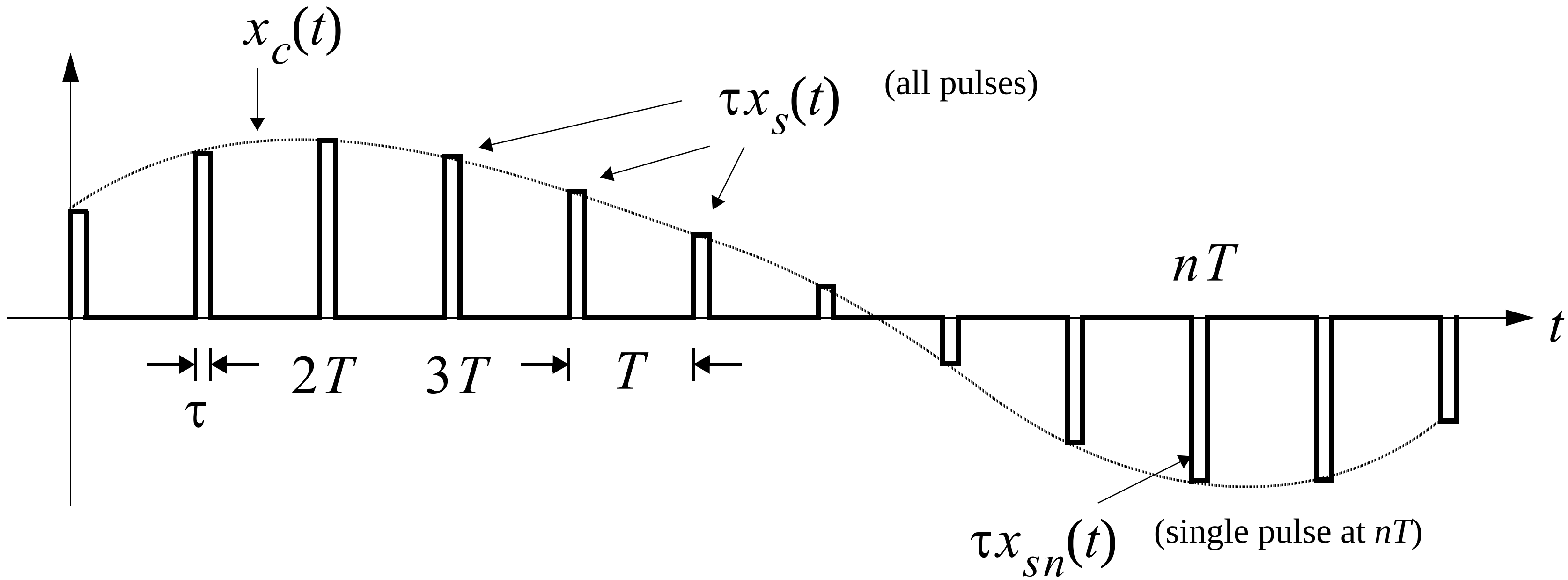
Example Signal Spectra

- $X_s(f)$ has same spectra as $X_c(f)$ but repeats every f_s (assuming no aliasing occurs).

- $X(\omega)$ has same spectra as $X_s(f)$ freq axis normalized.

- Spectra for $X_{sh}(f)$ equals $X_s(f)$ multiplied by $\frac{\sin x}{x}$ response — in effect, filtering out high frequency images.

Laplace Transform & Discrete-Time



- $x_s(t)$ scaled by τ such that the area under the pulse at nT equals the value of $x_c(nT)$.
- In other words, at $t = nT$, we have

$$x_s(nT) = \frac{x_c(nT)}{\tau} \tag{1}$$

Laplace Transform & Discrete-Time

- Thus as $\tau \to 0$, height of $x_s(t)$ at time nT goes to ∞ and so we plot $\tau x_s(t)$ instead.
- Define $\vartheta(t)$ to be the step function,

$$\vartheta(t) \equiv \begin{cases} 1 & (t \ge 0) \\ 0 & (t < 0) \end{cases} \qquad (2)$$

- then single-pulse signal, $x_{sn}(t)$, can be written as

$$x_{sn}(t) = \frac{x_c(nT)}{\tau}[\vartheta(t - nT) - \vartheta(t - nT - \tau)] \qquad (3)$$

and the entire signal $x_s(t)$ as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_{sn}(t) \qquad (4)$$

Laplace Transform & Discrete-Time

- Above signals are defined for ***all time*** — we can find Laplace transforms of these signals.
- The Laplace transform $X_{sn}(s)$ for $x_{sn}(t)$ is

$$X_{sn}(s) = \frac{1}{\tau}\left(\frac{1-e^{-s\tau}}{s}\right)x_c(nT)e^{-snT} \qquad (5)$$

and $X(s)$ is simply a linear combination of $x_{sn}(t)$, which results in

$$X_s(s) = \frac{1}{\tau}\left(\frac{1-e^{-s\tau}}{s}\right)\sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT} \qquad (6)$$

Laplace Transform & Discrete-Time

- Using the expansion $e^x = 1 + x + \frac{x^2}{2!} + \ldots$, when $\tau \to 0$, the term before the summation in (6) goes to unity.
- Therefore, as $\tau \to 0$,

$$X_s(s) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT} \tag{7}$$

- This Laplace transform only depends on sample points, $x_c(nT)$ which in turn depends on the ***relative*** sampling-rate, T.

Spectra of Discrete-Time Signals

- $x_s(t)$ spectra can be found by replacing $s = j\omega$ in (7)
- However, a more intuitive approach is ...
- Define a periodic pulse train, $s(t)$ as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad (8)$$

where $\delta(t)$ is the unit impulse function.

- Then $x_s(t)$ can be written as

$$x_s(t) = x_c(t)s(t) \qquad (9)$$

$$X_s(j\omega) = \frac{1}{2\pi} X_c(j\omega) \otimes S(j\omega) \qquad (10)$$

where $\otimes$ denotes convolution.

Spectra of Discrete-Time Signals

- Since the Fourier transform of a periodic impulse train is another periodic impulse train we have

$$S(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) \quad (11)$$

- Thus, the spectra $X_s(j\omega)$ is found to be

$$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\omega - \frac{jk2\pi}{T}\right) \quad (12)$$

- or equivalently,

$$X_s(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j2\pi f - jk2\pi f_s) \quad (13)$$

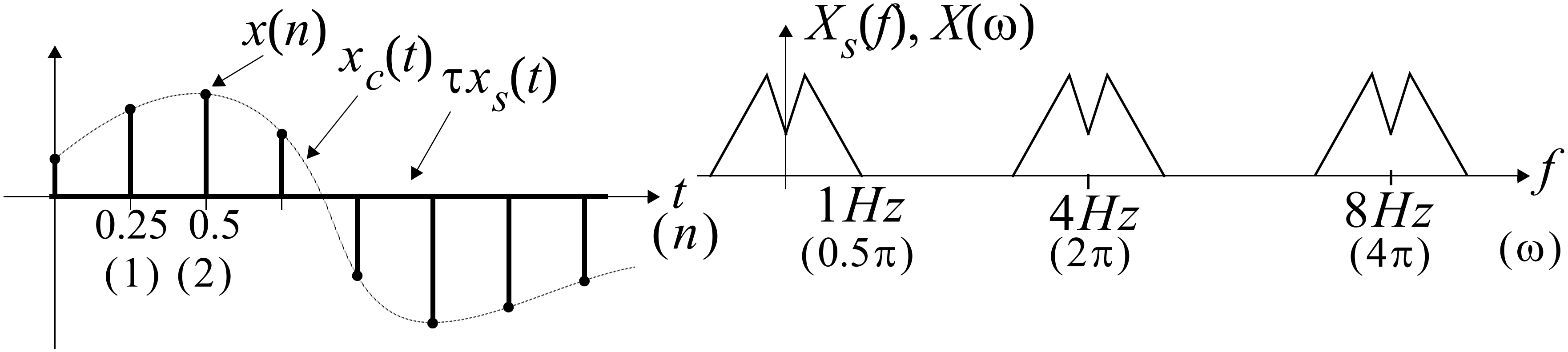
Spectra for Discrete-Time Signals

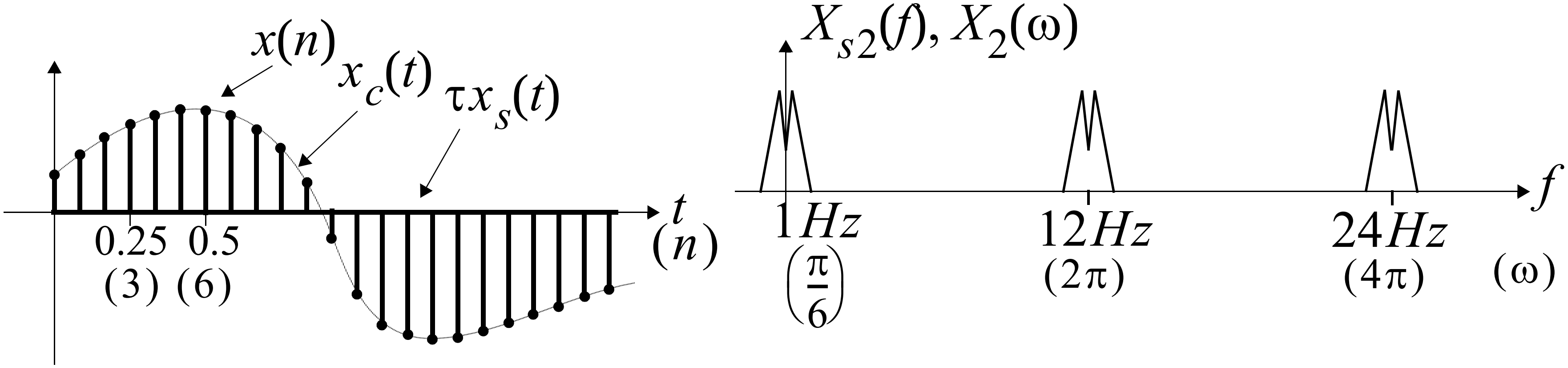
- The spectra for the sampled signal $x_s(t)$ equals a sum of shifted spectra of $x_c(t)$.

- ***No aliasing*** will occur if $X_c(j\omega)$ is bandlimited to $f_s/2$.

- Note that $x_s(t)$ can not exist is practice as it would require an infinite amount of power (seen by integrating $X_s(f)$ over all frequencies).

Spectra Example





Z-Transform

- The z-transform is merely a shorthand notation for (7).
- Specifically, defining

$$z \equiv e^{sT} \qquad (14)$$

- we can write

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x_c(nT) z^{-n} \qquad (15)$$

- where $X(z)$ is called the z-transform of the samples $x_c(nT)$.

Z-Transform

- 2 properties of the z-transform are:

  — If $x(n) \leftrightarrow X(z)$, then $x(n-k) \leftrightarrow z^{-k}X(z)$

  — Convolution in the time domain is equivalent to multiplication in the frequency domain.

$X(z)$ **is *not a function of the sampling-rate!***

- A 1Hz signal sampled at 10Hz has the same transform as a similar 1kHz signal sampled at 10kHz
- $X(z)$ is only related to the numbers, $x_c(nT)$ while $X_s(s)$ is the Laplace transform of the signal $x_s(t)$ as $\tau \rightarrow 0$.
- Think of the series of numbers as having a sample-rate normalized to $T = 1$ (i.e. $f_s = 1Hz$).

Z-Transform

- Such a normalization results in

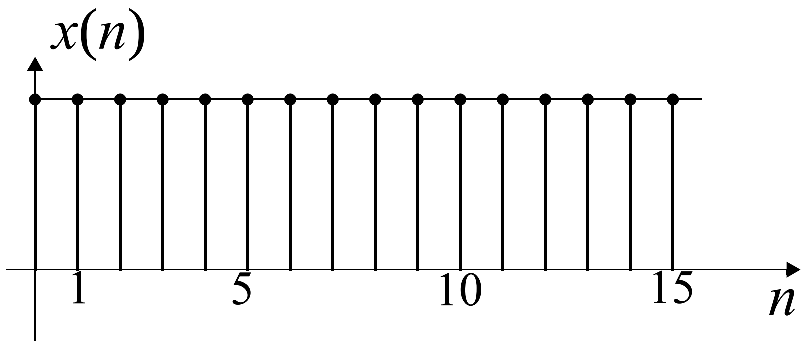
$$X_s(f) = X\left(\frac{2\pi f}{f_s}\right) \quad (16)$$

or equivalently, a frequency scaling of

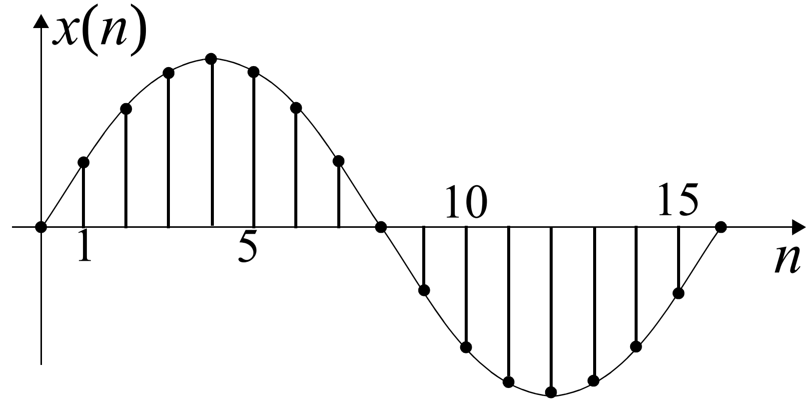
$$\omega = \frac{2\pi f}{f_s} \quad (17)$$

- ***Thus, discrete-time signals have ω in units of radians/sample***.
- Continuous-time signals have frequency units of cycles/second (hertz) or radians/second.

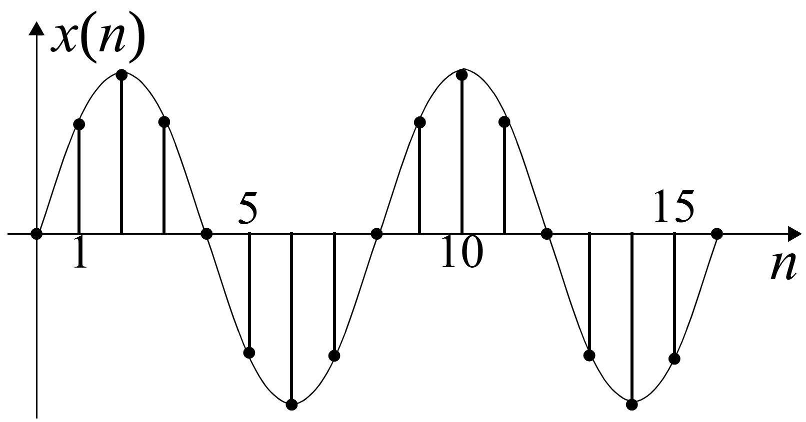
Example Sinusoidal Signals



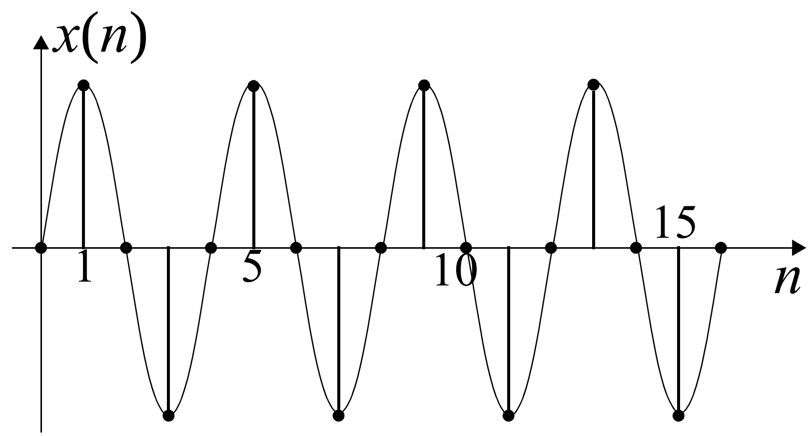
0 rad/sample = 0 cycles/sample



$\pi/8$ rad/sample = $1/16$ cycles/sample



$\pi/4$ rad/sample = $1/8$ cycles/sample

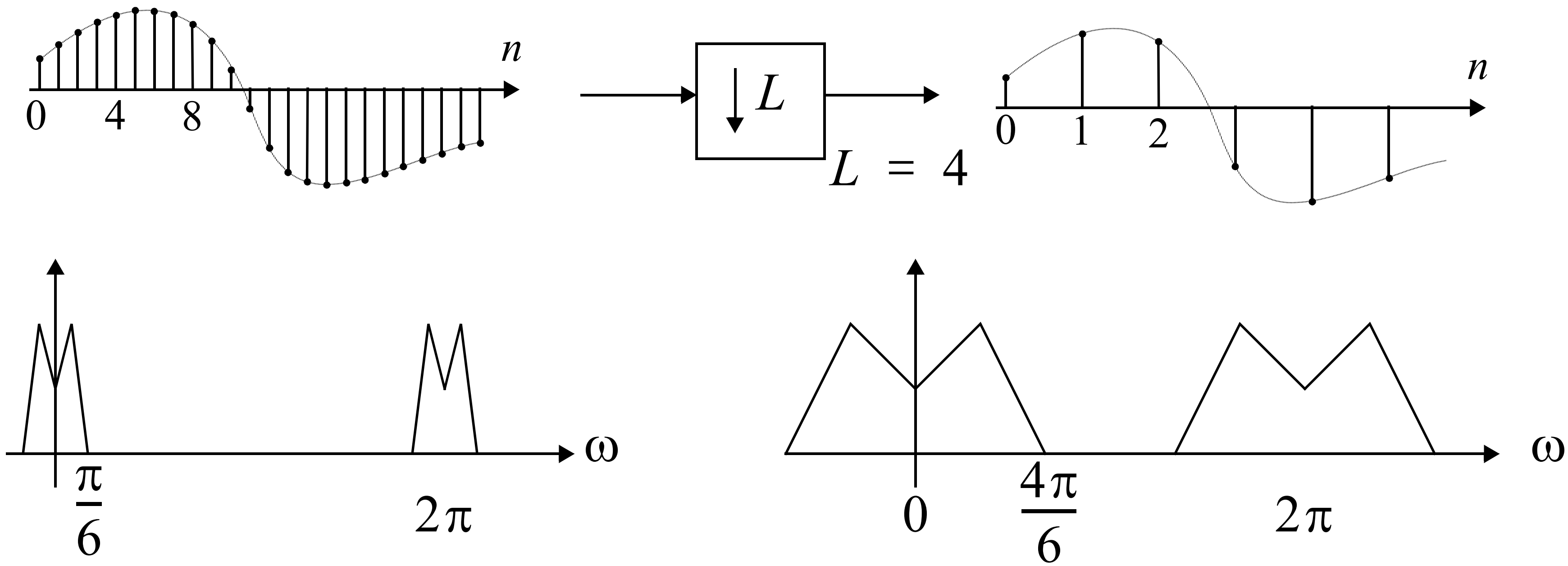


$\pi/2$ rad/sample = $1/4$ cycles/sample

Example Sinusoidal Signals

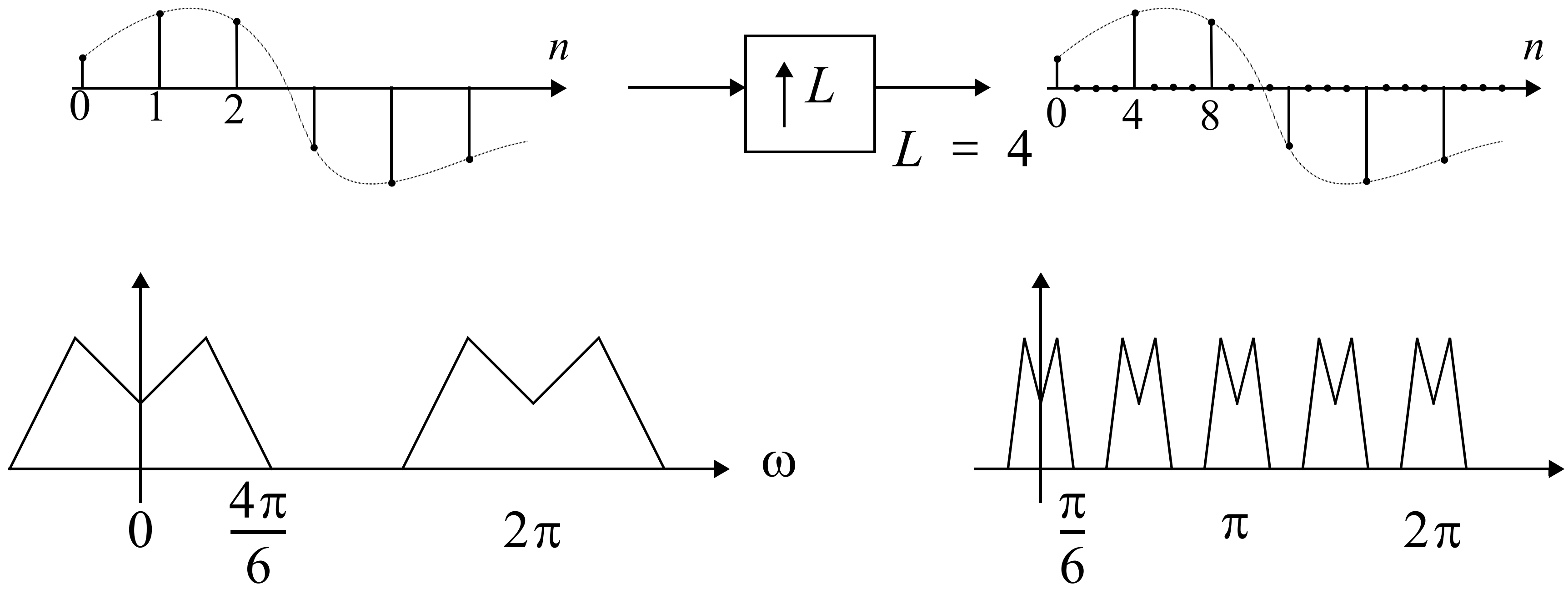
- A continuous-time sinusoidal signal of $1kHz$ when sampled at $4kHz$ will change by $\pi/2$ radians between each sample.
- Such a discrete-time signal is defined to have a frequency of $\pi/2$ rad/sample.
- Note that discrete-time signals are not unique since the addition of 2π will result in the same signal.
- For example, a discrete-time signal having a frequency of $\pi/4$ rad/sample is identical to that of $9\pi/4$ rad/sample.
- Normally discrete-time signals are defined to have frequency components only between $-\pi$ and π rad/sample.

Downsampling



- Keep every L‘th sample and throw away $L-1$ samples.
- It expands the original spectra by L.
- For aliasing not to occur, original signal must be bandlimited to π / L.

Upsampling



- Insert $L-1$ zero values between samples
- The frequency axis is scaled by L such that 2π now occurs where $L2\pi$ occurred in the original signal.
- No worry about aliasing here.

Discrete-Time Filters

$$u(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$$

(discrete-time filter)

($y(n)$ equals $h(n)$ if $u(n)$ is an impulse)

- An input series of numbers is applied to a discrete-time filter to create an output series of numbers.
- This filtering of discrete-time signals is most easily visualized with the shorthand notation of z-transforms.

Transfer-Functions

- Similar to those for continuous-time filters except instead of polynomials in “s”, polynomials in “z” are obtained.

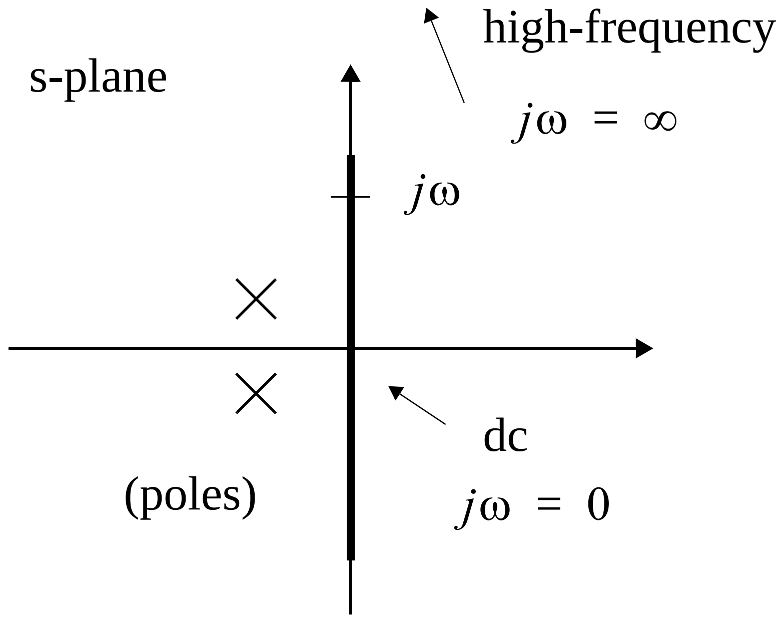
Cont-Time Transfer-Function

- Low-pass continuous-time filter, $H_c(s)$,

$$H_c(s) = \frac{4}{s^2 + 2s + 4} \tag{18}$$

- The poles are the roots of the denominator polynomial
- Poles: $-1.0 \pm 1.7321j$ for this example.
- Zeros: Defined to have two zeros at ∞ since the den poly is two orders higher than the numerator poly.

Cont-Time Frequency Response



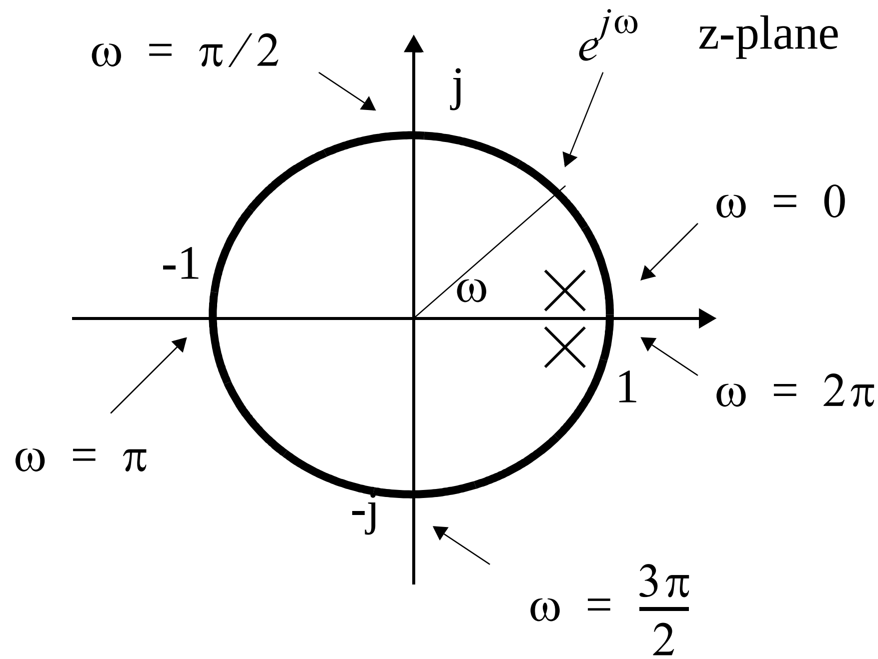
- Poles and zeros plotted in the s-plane.
- Substitution $s = j\omega$ is equivalent to finding the magnitude and phase of vectors from a point along the $j\omega$ axis to all the poles and zeros.

Discrete-Time Transfer-Function

$$H(z) = \frac{0.05}{z^2 - 1.6z + 0.65} \tag{19}$$

- Poles: $0.8 \pm 0.1j$ in the z-plane and two zeros are again at ∞.
- To find the frequency response of $H(z)$, the poles and zeros can be plotted in the z-plane, and the unit circle contour is used, $z = e^{j\omega}$

Discrete-Time Frequency Response

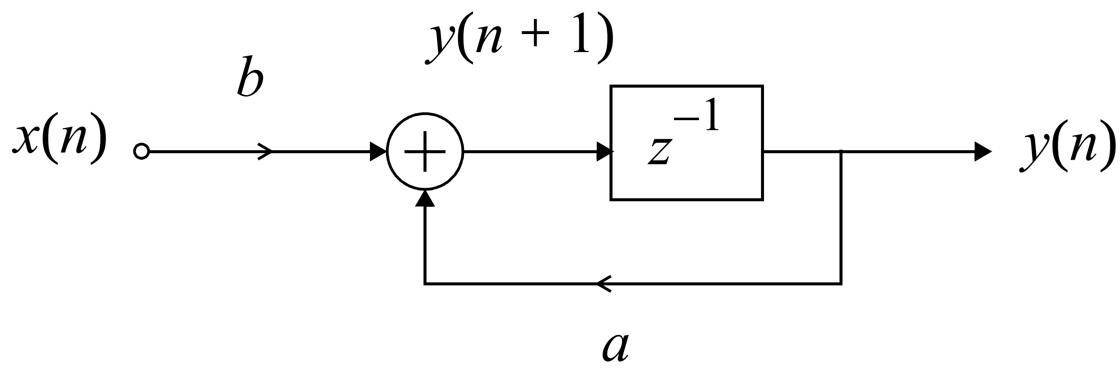


- Note that poles or zeros occurring at $z = 0$ do not affect the magnitude response of $H(z)$ since a vector from the origin to the unit circle always has a length of unity. However, they would affect the phase response.

Discrete-Time Frequency Response

- $z = 1$ corresponds to the frequency response at both dc (i.e. $\omega = 0$) and for $\omega = 2\pi$.
- The time normalization of setting $T = 1$ implies that $\omega = 2\pi$ is equivalent to the sampling-rate speed (i.e. $f = f_s$) for $X_s(f)$.
- As with cont-time filter, if filter coefficients are real, poles and zeros occur in complex-conjugate pairs — magnitude is symmetric, phase is anti-symmetric.
- Going around the circle again would give the same result as the first time implying that the frequency response repeats every 2π.

Stability of Discrete-Time Filters



- To realize rational polynomials in "z", discrete-time filters use delay elements (i.e. "z^{-1}" building blocks) much the same way that analog filters can be formed using integrators (i.e. "s^{-1}" building blocks).
- The result is finite difference equations describing discrete-time filters

Stability of Discrete-Time Filters

- A finite difference equation can be written for above system

$$y(n+1) = bx(n) + ay(n) \tag{20}$$

- In the z-domain, this equation is written as

$$zY(z) = bX(z) + aY(z) \tag{21}$$

- We find $H(z)$ given by

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{b}{z-a} \tag{22}$$

which has a pole on the real axis at $z = a$.

Stability of Discrete-Time Filters

- To test for stability, let the input $x(n)$ be an impulse

$$y(0) = k$$

where k is some arbitrary initial state value for y.

$$y(1) = b + ak$$
$$y(2) = ab + a^2 k$$
$$y(3) = a^2 b + a^3 k$$
$$y(4) = a^3 b + a^4 k$$
$$\vdots$$

Stability of Discrete-Time Filters

- The response, $h(n)$, is seen to be given by

$$h(n) = \begin{cases} 0 & (n < 1) \\ (a^{n-1}b + a^{n}k) & (n \ge 1) \end{cases} \qquad (23)$$

- This response remains bounded only when $|a| \le 1$ for this first-order filter and is unbounded otherwise.

- ***In general, a linear time-invariant discrete-time filter, $H(z)$, is stable if and only if all its poles are located within the unit circle.***

IIR Filters

- Infinite-Impulse-Response (IIR) filters are those discrete-time filters that when excited by an impulse, their outputs remain non-zero assuming infinite precision arithmetic.
- The above example is IIR when $a \neq 0$
- IIR filters can be more efficient when long impulse responses are needed.
- They have some unusual behaviors due to finite-precision effects such as limit-cycles.

FIR Filters

- Finite-Impulse-Response (FIR) filters are those discrete-time filters that when excited by an impulse, their outputs go precisely to zero (and remain zero) after a finite value of n.
- Example — running average of 3

$$y(n) = \frac{1}{3}(x(n) + x(n-1) + x(n-2)) \tag{24}$$

$$H(z) = \frac{1}{3}\sum_{i=0}^{2} z^{-i} \tag{25}$$

- Has poles but they all occur at $z = 0$.
- FIR filters are always stable and exact linear phase filters can be realized.

Bilinear Transform

- Consider $H_c(p)$ as a continuous-time transfer-function (where "p" is the complex variable equal to $\sigma_p + j\Omega$), the bilinear transform is defined to be given by,

$$p = \frac{z-1}{z+1} \tag{26}$$

- The inverse transformation is given by,

$$z = \frac{1+p}{1-p} \tag{27}$$

- The z-plane locations of 1 and -1 (i.e. dc and $f_s/2$) are mapped to p-plane locations of 0 and ∞, respectively.

Bilinear Transform

- The unit circle, $z = e^{j\omega}$, in the z-plane is mapped to the entire $j\Omega$ axis in the p-plane.

$$p = \frac{e^{j\omega} - 1}{e^{j\omega} + 1} = \frac{e^{j(\omega/2)}(e^{j(\omega/2)} - e^{-j(\omega/2)})}{e^{j(\omega/2)}(e^{j(\omega/2)} + e^{-j(\omega/2)})} \tag{28}$$

$$= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2) \tag{29}$$

- Results in the following frequency “warping”.

$$\Omega = \tan(\omega/2) \tag{30}$$

Bilinear Transform Filter Design

- Design a continuous-time transfer-function, $H_c(p)$, and choose the discrete-time transfer-function, $H(z)$, such that

$$H(z) \equiv H_c((z-1)/(z+1)) \qquad (31)$$

so that

$$H(e^{j\omega}) = H_c(j\tan(\omega/2)) \qquad (32)$$

- The response of $H(z)$ is seen to be equal to the response of $H_c(p)$ except with a frequency "warping"
- Order of the cont-time and discrete-time also same.

Bilinear Design Example

- Find a first-order $H(z)$ that has a 3db frequency at $f_s/20$, a zero at -1 and a dc gain of one.
- Using (30), the frequency value, $f_s/20$, or equivalently, $\omega = (2\pi)/20 = 0.314159$ is mapped to $\Omega = 0.1584$.
- Thus, $H_c(p)$ should have a 3dB frequency value of 0.1584 rad/s.
- Such a 3db frequency value is obtained by having a p-plane zero equal to ∞ and pole equal to -0.1584.

Bilinear Design Example

- Transforming these continuous-time pole and zero back using (27) results in a z-plane zero at -1 and a pole at 0.7265.
- Therefore, $H(z)$ appears as

$$H(z) = \frac{k(z+1)}{z-0.7265} \qquad (33)$$

- The constant k can be determined by setting the dc gain to one, or equivalently, $|H(1)| = 1$ which results in $k = 0.1368$.

Sample-and-Hold Response

- A sample-and-held signal, $x_{sh}(t)$, is related to its sampled signal by the mathematical relationship,

$$x_{sh}(t) = \sum_{n=-\infty}^{\infty} x_c(nT)[\vartheta(t-nT) - \vartheta(t-nT-T)] \tag{34}$$

- $x_{sh}(t)$ is well-defined for all time and thus the Laplace transform can be found to be equal to

$$X_{sh}(s) = \frac{1-e^{-sT}}{s} \sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT}$$

$$= \frac{1-e^{sT}}{s} X_s(s) \tag{35}$$

Sample-and-Hold Response

- The hold transfer-function, $H_{sh}(s)$, is equal to

$$H_{sh}(s) = \frac{1 - e^{-sT}}{s} \tag{36}$$

- The spectra for $H_{sh}(s)$ is found by substituting $s = j\omega$

$$H_{sh}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T \times e^{-\frac{j\omega T}{2}} \times \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \tag{37}$$

Sample-and-Hold Response

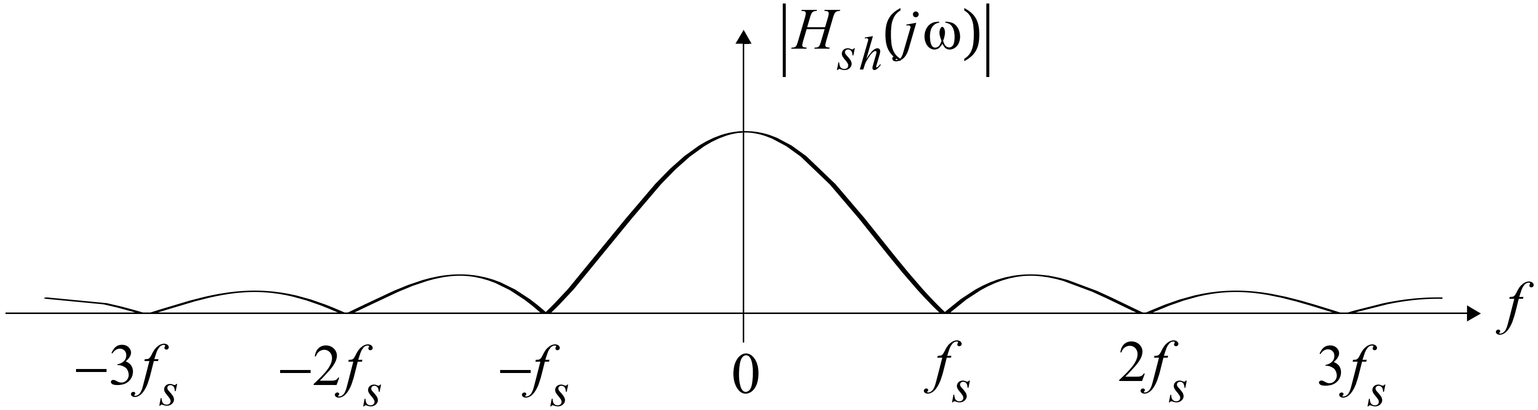
- The magnitude of this response is given by

$$\left|H_{sh}(j\omega)\right| = T\frac{\left|\sin\left(\frac{\omega T}{2}\right)\right|}{\left|\frac{\omega T}{2}\right|}$$

$$\text{or} \quad \left|H_{sh}(f)\right| = T\frac{\left|\sin\left(\frac{\pi f}{f_s}\right)\right|}{\left|\frac{\pi f}{f_s}\right|} \qquad (38)$$

- and is often referred to as the “$\frac{\sin x}{x}$” or “sinc” response.

Sample-and-Hold Response



- This frequency shaping of a sample-and-hold only occurs for a continuous-time signal.
- Specifically, a sample-and-hold before an A/D converter does not aid in any anti-aliasing requirement since the A/D converter has a true discrete-time output.