Chapter 9 - Problems

9.1) Show that $x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z)$

The z-transform for $x(n)$ is

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad \text{for } z \triangleq e^{sT}$$

$$= \ldots + x(-1)z^{-1} + x(0)z^{0} + x(1)z^{1} + \ldots$$

Given that $x_2(n) = x(n-k)$,

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x(n-k)\, z^{-n} \quad \boxed{1}$$

Let $i = n-k$

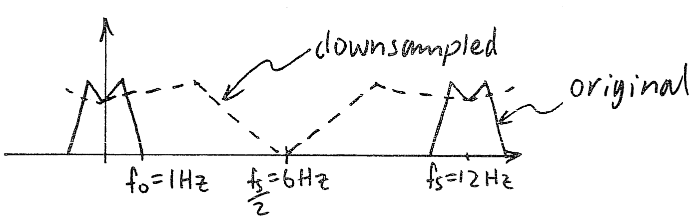
$\therefore$ $n = i+k$ and we can rewrite $\boxed{1}$ as

$$X_2(z) = \sum_{i=-\infty}^{\infty} x(i)\, z^{-(i+k)} = z^{-k} \underbrace{\sum_{i=-\infty}^{\infty} x(i)\, z^{-i}}_{=X(z)}$$

$\therefore$ $X_2(z) = z^{-k}X(z)$

Q.E.D.

9.2)



$\because$ bandwidth is 1 Hz, a min. sampling rate of 2 Hz is required.

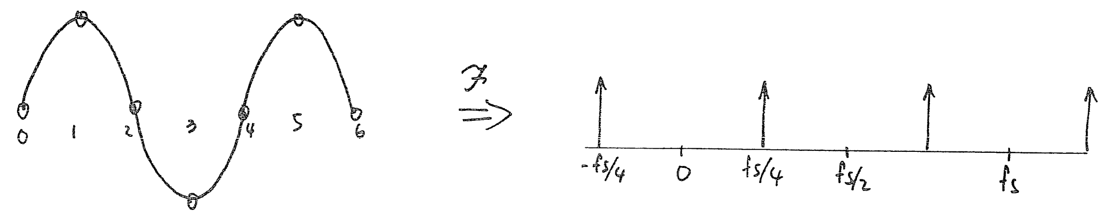
$\therefore$ The signal can be downsampled by a factor of

$$\frac{f_s}{f_{min}} = \frac{12\text{Hz}}{2\text{Hz}} = 6.$$

9.3) Downsampling is not time-invariant.

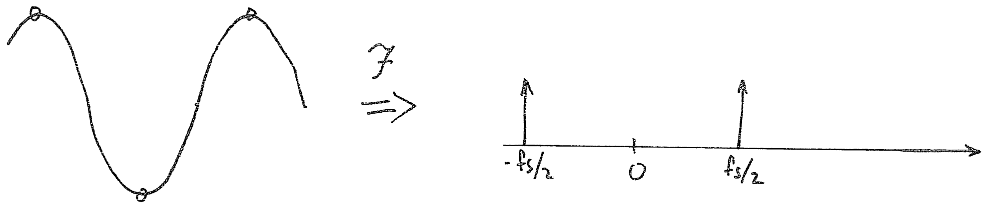
Counterexample:

Given the sampled sinusoid and its spectrum:



Consider the case where we downsample by two, done simply by discarding every other sample.

If we discard all the even samples we get, as expected,



However, by discarding all the odd samples, we are left with a constant vector of zeros, and effectively no signal.

As for linearity, downsampling is linear $\Leftrightarrow$

Given $w(n) = a x(n) + b y(n)$,
its downsampled equivalent, $w'(n)$, is given by
$$w'(n) = a\,x'(n) + b\,y'(n)$$
where $x'(n)$ and $y'(n)$ are equivalently downsampled equivalents of $x(n)$ and $y(n)$.

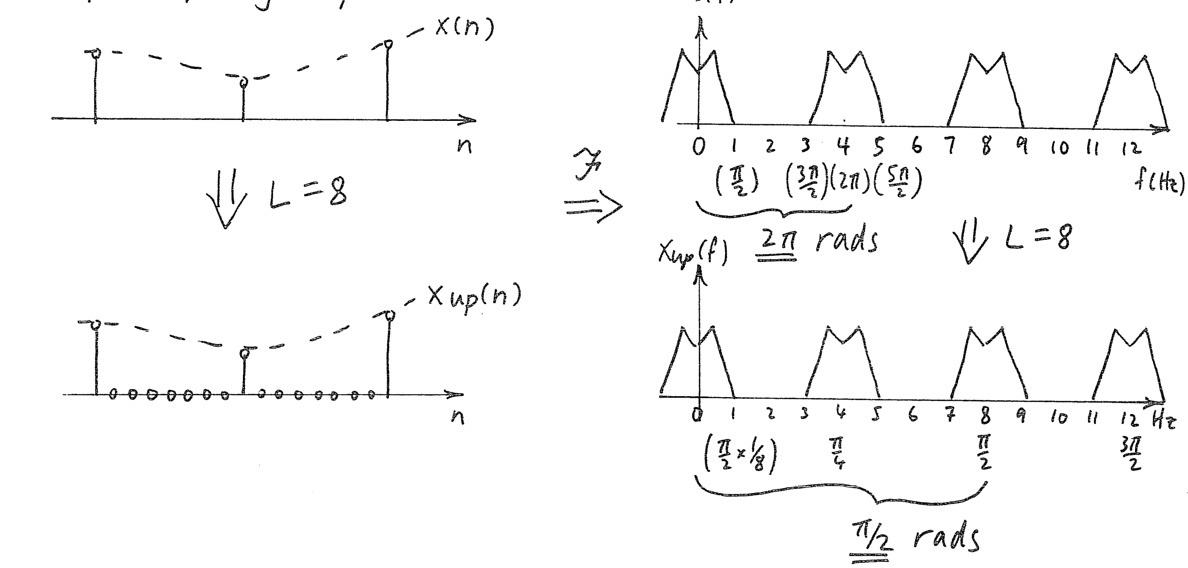
Proof: For downsampling factor of L

$$w'(n) = w(nL+k) \qquad n, L \in I,\ k \in [0, L-1]$$
$$= a\,\underbrace{x(nL+k)} + b\,\underbrace{y(nL+k)}$$

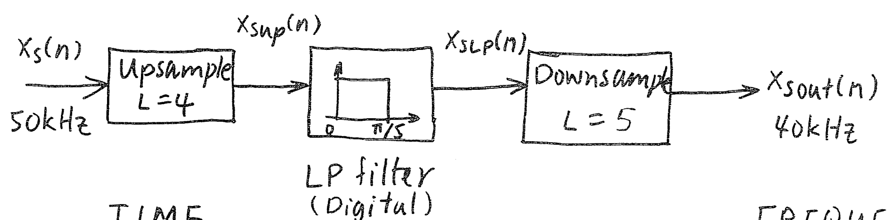
But these are simply downsampled versions of $x(n)$ and $y(n)$, $x'(n)$ and $y'(n)$ respectively.

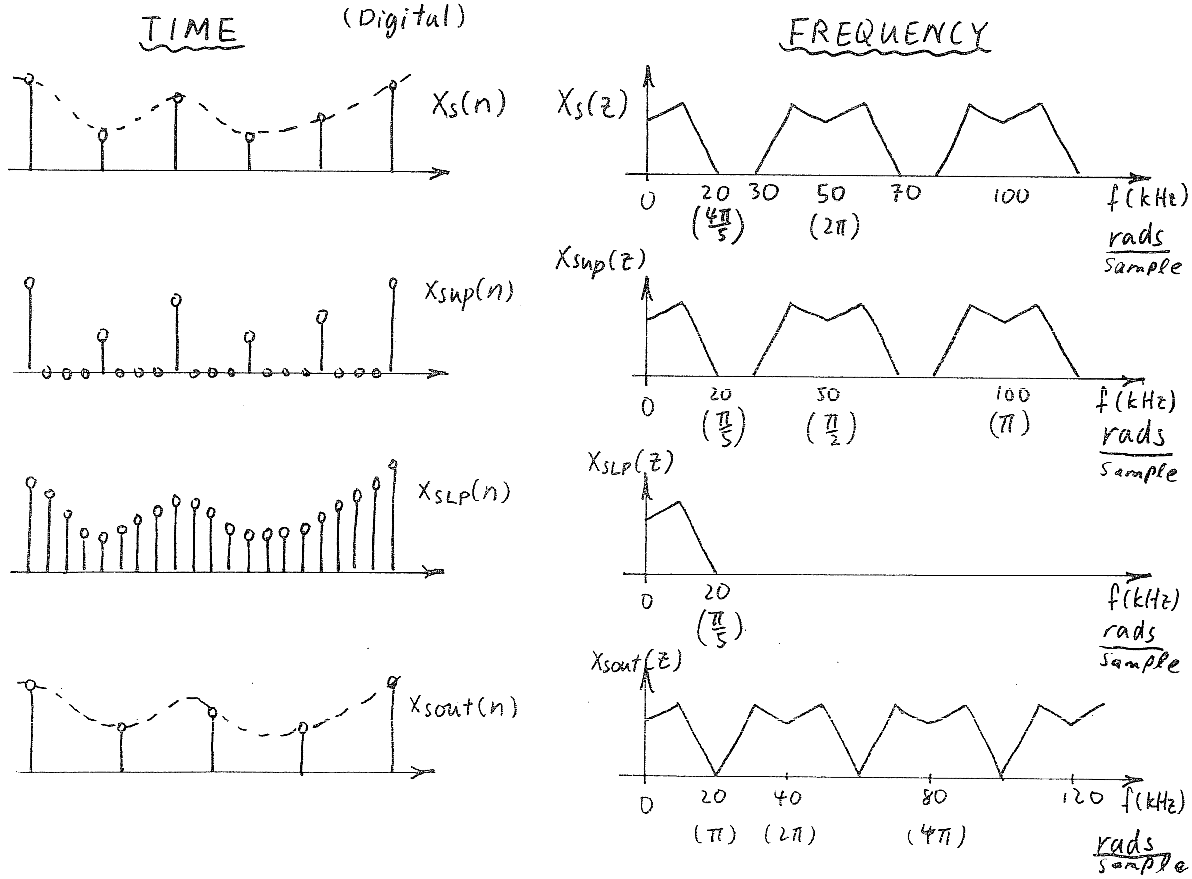
$\therefore w'(n) = a\,x'(n) + b\,y'(n)$ Q.E.D.

9.4) Upsampling by 8 :



9.5) Block diagram of sample rate converter :





9.6) Find poles and zeros of

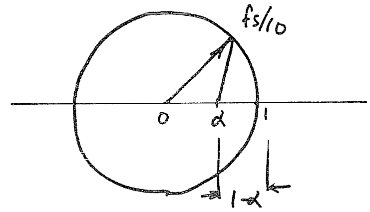
$$H(z) = \frac{0.05}{z^2 - 1.6z + 0.65}$$

For poles ; $\quad z_p^2 - 1.6 z_p + 0.65 = 0$

$$\therefore \quad z_p = 0.8 \pm i\,0.1 = 0.806 \angle 7^\circ$$

By inspection, 2 zeros are at $z_z = \infty$.

9.7) Given $\quad H(z) = \frac{K}{z - \alpha}, \quad K, \alpha \in \mathbb{R}$



Find α such that $\quad f_{-3dB} = fs/10$.

At DC or $z = 1$

$$|H(z)|_{z=1} = \frac{K}{1-\alpha} \equiv |H(\omega)|\Big|_{\omega=0} \qquad \boxed{1}$$

Now $\quad f_{-3dB} = fs/10 \Rightarrow \omega_{-3dB} = \frac{2\pi}{10} = \pi/5$ rads/sec

$$\therefore \quad |H(\omega_{-3dB})| = |H(z)|_{z = e^{j\pi/5}}$$

$$= \frac{K}{|e^{j\pi/5} - \alpha|} \qquad \boxed{2}$$

Now set $\quad |H(\omega_{-3dB})| = \frac{1}{\sqrt{2}} |H(z)|_{z=1}$

$$\therefore \quad \frac{\cancel{K}}{|e^{j\pi/5} - \alpha|} = \frac{1}{\sqrt{2}} \frac{\cancel{K}}{1-\alpha}$$

$$|e^{j\pi/5} - \alpha| = \sqrt{2}(1-\alpha)$$

$$\sqrt{(\cos(\pi/5) - \alpha)^2 + \sin^2(\pi/5)} = \sqrt{2}(1-\alpha)$$

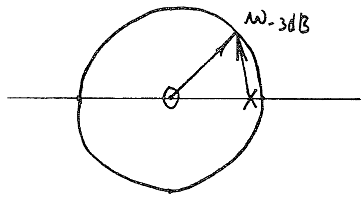
$$\cos^2(\pi/5) - 2\alpha\cos(\pi/5) + \alpha^2 + \sin^2(\pi/5) = 2\alpha^2 - 4\alpha + 2$$

$$0 = \alpha^2 + \alpha(-4 + 2\cos(\pi/5)) + 2 - 1$$

$$0 = \alpha^2 - 2.382\alpha + 1$$

$\therefore \quad \alpha = \underset{\text{unstable root}}{\underline{1.837}}$ or $\boxed{0.544}$ $\qquad \therefore$ the pole is located at $z = 0.544$

9.8) Sketch $|H(z)|$ given $H(z) = \dfrac{0.05z}{z-0.95}$



$\therefore$ the zero is located at $z=0$, it does not affect the magnitude response

$$|H(z)|\Big|_{z=e^{j0}} = \frac{0.05}{0.05} = \underline{0\,dB}$$

$$|H(z)|\Big|_{z=e^{j\pi}} = \frac{-0.05}{-1.05} = \underline{-26.4\,dB}$$

For ω_{-3dB},

$$|H(\omega_{-3dB})| = \frac{1}{\sqrt{2}}|H(z)|\Big|_{z=e^{j0}}$$

$$\frac{0.05}{|e^{j\omega_{-3dB}}-0.95|} = \frac{1}{\sqrt{2}}$$

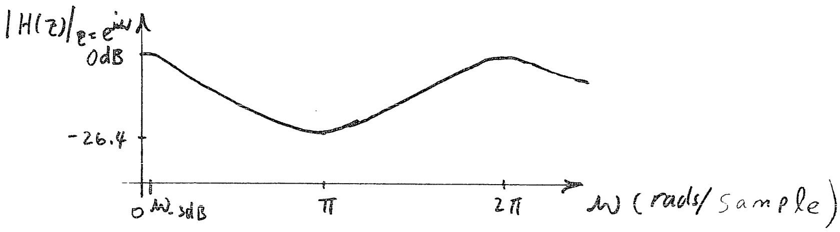
$$(\sqrt{2}\times 0.05)^2 = ((\cos(\omega_{-3dB})-0.95)^2+\sin^2(\omega_{-3dB}))$$

$$0.005 = 1-1.9\cos(\omega_{-3dB})+0.95^2$$

$$\cos\omega_{-3dB} = 0.99868$$

$$\therefore \underline{\omega_{-3dB} = 0.051 \text{ rads/sample}}$$

And the frequency response is



9.9) Given $y(n) = u(n) - 0.3\,y(n-1)$, determine the impulse response.

Now $u(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$

$y(0) = 0$

$y(1) = u(1) = 1$

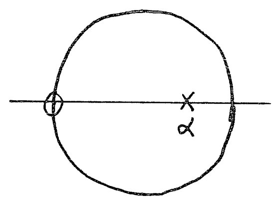
$y(2) = -0.3\,y(1) = -0.3$

$y(3) = -0.3\,y(2) = (-0.3)^2$

$\vdots$

$\therefore\ y(n) = \begin{cases} 0 & n = 0 \\ (-0.3)^{n-1} & n > 0 \end{cases}$ Impulse response

9.10) Find $H(z)$. DC gain $= 1$



From the basic information, we know that

$$H(z) = k \times \frac{z+1}{z-\alpha}$$

For k, $\|H(z)\|\big|_{z=1} = k\,\frac{2}{1-\alpha} \equiv 1$

$$\therefore\ k = \frac{1-\alpha}{2}$$

For α, $|H(z)|\big|_{z=e^{j\omega_{3dB}}} = \frac{1}{\sqrt{2}}\,|H(z)|\big|_{z=e^{j0}}$ where $\omega_{3dB} = 2\pi/30 = \pi/15$

$$\frac{1-\alpha}{2} \times \frac{|e^{j\pi/15}+1|}{|e^{j\pi/15}-\alpha|} = \frac{1}{\sqrt{2}}$$

$$\frac{2}{(1-\alpha)^2} = \frac{(\cos\pi/15 + 1)^2 + \sin^2\pi/15}{(\cos\pi/15 - \alpha)^2 + \sin^2\pi/15} = \frac{3.9563}{\alpha^2 - 1.9563\alpha + 1}$$

$$0 = (3.9563-2)\alpha^2 + 2\times(1.9563-3.9563)\alpha + (3.9563-2)$$

$$0 = \alpha^2 - 2.0447\alpha + 1$$

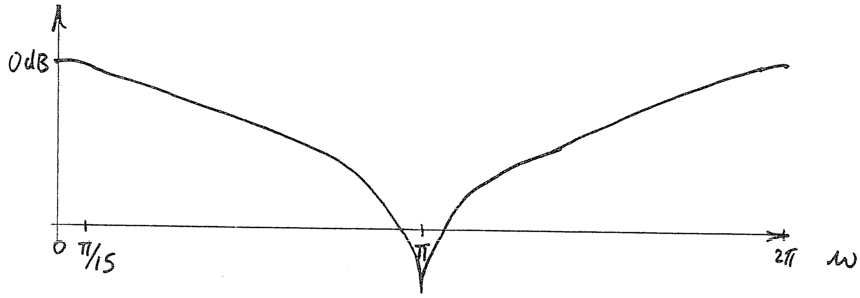
$$\therefore\ \alpha = 0.8098$$

and $k = \frac{1-0.8098}{2} = 0.095$

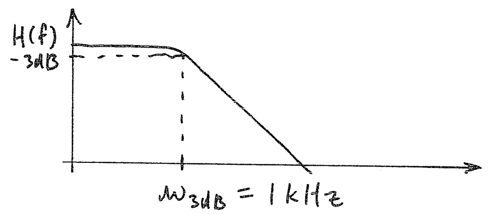
(cont.)

9.10) (cont.)

$$\therefore\ H(z) = \frac{0.095(z+1)}{z-0.8098}$$



9.11)



$f_s = 100\,kHz$
$Q = 1/\sqrt{2}$

$\omega_{3dB} = 1\,kHz$ corresponds to $\frac{2\pi}{100} = \pi/50$ rads/sample in the z-plane

The design is done in the p-plane where ω_{3dB} is mapped to

$$\Omega_{3dB} = \tan(\pi/50 \times 1/2)$$

$$\Omega_{3dB} = 0.03143 \text{ rads/sec}$$

$$\therefore\ H_p(p) = \frac{\Omega_{3dB}^2}{p^2 + \frac{\Omega_{3dB}}{Q}p + \Omega_{3dB}^2} = \frac{0.03143^2}{p^2 + \frac{0.03143}{1/\sqrt{2}}p + 0.03143^2}$$

$$= \frac{9.88\times10^{-4}}{p^2 + 4.44\times10^{-2}p + 9.88\times10^{-4}}$$

Convert this function to the z-plane using $p = \frac{z-1}{z+1}$

$$\therefore\ H(z) = 9.88\times10^{-4}\left[\left(\frac{z-1}{z+1}\right)^2 + 4.44\times10^{-2}\left(\frac{z-1}{z+1}\right) + 9.88\times10^{-4}\right]^{-1}$$

$$= \frac{9.88\times10^{-4}(z+1)^2}{(1+9.88\times10^{-4}+4.44\times10^{-2})z^2 + (2\times9.88\times10^{-4}-2)z + (1-4.44\times10^{-2}+9.88\times10^{-4})}$$

$$\therefore\ H(z) = \frac{9.45\times10^{-4}(z+1)^2}{z^2 - 1.91z + 0.915}$$

9.12) Given the integrator transfer function,

$$\frac{Y(z)}{X(z)} = \frac{1}{z-1}$$

determine the output to a sinusoidal input.

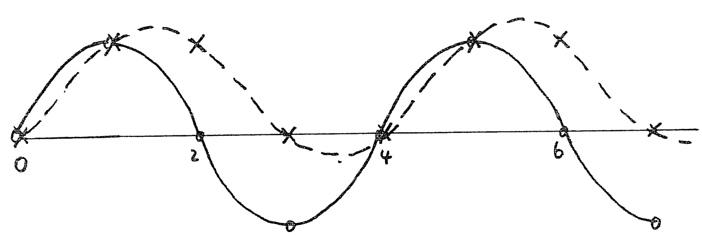
$$\frac{Y(z)}{X(z)} = \frac{1}{z-1}$$

$$\therefore \quad zY(z) - Y(z) = X(z)$$

$$\Rightarrow \quad y(n+1) = x(n) + y(n)$$

Assume $y(0) = 0$

$$\begin{aligned}
\therefore \quad y(1) &= x(0) + y(0) = 1 + 0 = 1 \\
y(2) &= 0 + 1 = 1 \\
y(3) &= -1 + 1 = 0 \\
y(4) &= 0 + 0 = 0 \\
y(5) &= 1 + 0 = 1 \\
y(6) &= 0 + 1 = 1 \\
y(7) &= -1 + 1 = 0 \quad \ldots
\end{aligned}$$



Time Domain

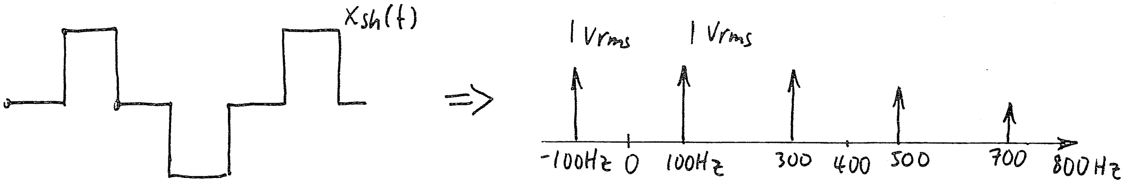
$$H(z)\Big|_{z=e^{j\pi/2}} = \frac{1}{j+1} = \frac{1}{\sqrt{2}}\angle{-45^\circ}$$

frequency domain

Interpolating the output samples, we see a phase shift of 45° and the amplitude looks about 3dB lower. Thus our time and frequency domain results are consistent.

The dc shift is due to the initial state being 0.

9.13)



$$H_{SH}(f)\Big|_{f=100Hz} = T\,\frac{|\sin(\pi f/f_s)|}{\pi f/f_s}\Big|_{f=100Hz} = T\cdot\frac{|\sin(\frac{100\pi}{400})|}{100\pi/400}$$

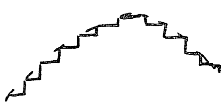
$$= \underline{0.9003\,T \equiv 1\,Vrms}$$

$$H_{SH}(f)\Big|_{f=300Hz} = T\,\frac{|\sin(3\pi/4)|}{3\pi/4} = 0.3001\,T = \underline{0.33\,Vrms}$$

$$H_{SH}(f)\Big|_{f=500Hz} = T\,\frac{|\sin(5\pi/4)|}{5\pi/4} = 0.1801\,T = \underline{0.2\,Vrms}$$

$$H_{SH}(f)\Big|_{f=700Hz} = T\,\frac{|\sin(7\pi/4)|}{7\pi/4} = 0.1286\,T = \underline{0.14\,Vrms}$$

9.14) 100Hz signal is now created with a sample rate of 10kHz



$\therefore$ We have 100 points / period

$$H_{SH}(f) = T\,\frac{|\sin(\pi f/f_s)|}{\pi f/f_s}$$

$$H_{SH}(f)\Big|_{f=100Hz} = T \times \frac{|\sin(100\pi/10000)|}{100\pi/10000} = \underline{0.9998\,T = 1\,Vrms}$$

For the images,

$$H_{SH}(f)\Big|_{f=9.9kHz} = T\,\frac{|\sin(9900\pi/10000)|}{9900\pi/10000} = 1.01\times10^{-2}\,T = \underline{10.1\,mVrms}$$

$$H_{SH}(f)\Big|_{f=10.1kHz} = T\,\frac{|\sin(10100\pi/10000)|}{10100\pi/10000} = 0.99\times10^{-2}\,T = \underline{9.9\,mVrms}$$

9.15) Repeat Example 9.8 with $f_{in} = 100\,Hz$

For $|H(z)|$,

$$z_{in} = e^{j(2\pi \times 100/50000)} = e^{j\pi/250} = 0.99992 \pm j0.01257$$

$$\therefore |H(z_{in})| = \frac{0.2}{|(0.99992 - 0.8) + j0.01257|} = \frac{0.2}{0.20032} = \underline{0.9984}$$

This is the same for the images.

For the sample-hold circuitry,

$$H_{SH}(f)\big|_{f=100Hz} = T\,\frac{|\sin(100\pi/50000)|}{100\pi/50000} = 0.99999T$$

$$H_{SH}(f)\big|_{f=99.9kHz} = T\,\frac{|\sin(99900\pi/50000)|}{99900\pi/50000} = 1.001 \times 10^{-3}\,T$$

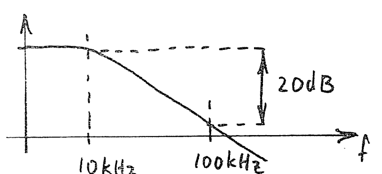
$$H_{SH}(f)\big|_{f=100.1kHz} = T\,\frac{|\sin(100100\pi/50000)|}{100100\pi/50000} = 9.99 \times 10^{-4}\,T$$

$$\therefore |y_{sh}(t)|\big|_{f=100Hz} = 0.9984 \times 0.99999\cancel{T} \times \frac{1}{\cancel{T}} \times 1.0\,Vrms = \underline{998\,mVrms}$$

$$|y_{sh}(t)|\big|_{f=99.9kHz} = 0.9984 \times 1.001 \times 10^{-3} \times 1\,Vrms = \underline{0.999\,mVrms}$$

$$|y_{sh}(t)|\big|_{f=100.1kHz} = 0.9984 \times 9.99 \times 10^{-4} \times 1\,Vrms = \underline{0.997\,mVrms}$$

9.16)



$$|H(\omega)|\big|_{\omega = 100kHz} = \frac{1}{|10j + 1|} = \frac{1}{10.05} = \underline{-20.0\,dB}$$

$\because$ 20 dB of attenuation is already provided by the microphone at 100 kHz.

$\therefore$ Another 60dB of attenuation is required from the anti-aliasing filter to obtain 80 dB noise rejection.