SOLVING THE CONVEX COST INTEGER DUAL NETWORK FLOW PROBLEM

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ABSTRACT

In this paper, we consider an integer convex optimization problem where the objective function is the sum of separable convex functions (that is, of the form $\sum_{(i,j)\in Q} \bar{F}_{ij}(w_{ij}) + \sum_{i\in P} \bar{B}_i(\mu_i)$), the constraints are similar to those arising in the dual of a minimum cost flow problem (that is, of the form $\mu_i - \mu_j \le w_{ij}$, $(i, j) \in Q$), with lower and upper bounds on variables. Let $n = |P|$, $m = |Q|$, and U be the largest magnitude in the lower and upper bounds of variables. We call this problem *the convex cost integer dual network flow problem*. In this paper, we describe several applications of the convex cost integer dual network flow problem arising in dial-a-ride transit problems, inverse spanning tree problem, project management, and regression analysis. We develop network flow based algorithms to solve the convex cost integer dual network flow problem. We show that using the Lagrangian relaxation technique, the convex cost integer dual network flow problem can be transformed to a convex cost primal network flow problem where each cost function is a piecewise linear convex function with integer slopes. Its special structure allows the convex cost primal network flow problem to be solved in O(nm log n log(nU)) time using a cost-scaling algorithm, which is the best available time bound to solve the convex cost integer dual network flow problem.

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1. INTRODUCTION

In this paper, we consider the following integer programming problem with convex costs:

$$\text{Minimize } \sum_{(i,\, j) \in Q} \bar{F}_{ij}(w_{ij}) + \quad \sum_{i \in P} \bar{B}_i(\mu_i) \qquad \text{(1a)}$$

subject to

$$\mu_i - \mu_j \le w_{ij} \qquad \text{for all } (i, j) \in Q, \qquad \text{(1b)}$$

$$l_{ij} \le w_{ij} \le u_{ij} \qquad \text{for all } (i, j) \in Q, \qquad \text{(1c)}$$

$$l_i \le \mu_i \le u_i \qquad \text{for all } i \in P, \qquad \text{(1d)}$$

$$w_{ij} \text{ is integer for all } (i, j) \in Q, \text{ and } \mu_i \text{ is integer for all } i \in P, \qquad \text{(1e)}$$

where $P = \{1, 2, \ldots, n\}$ is a set of n numbers, $Q \subseteq P \times P$, and l_i, u_i, l_{ij}, and u_{ij} are specified integers. Let $m = |Q|$. In this problem, μ_i's and w_{ij}'s are decision variables, $\bar{F}_{ij}(w_{ij})$ is a convex function of w_{ij} for every $(i, j) \in Q$, and $\bar{B}_i(\mu_i)$ is also a convex function of μ_i for every $i \in P$. Let $U = \max[\max\{u_{ij}, l_{ij}: (i, j) \in Q\}, \max\{u_i, l_i : i \in P\}]$. We assume that $U < \infty$.

We call problem (1) the *convex cost integer dual network flow problem* (or simply the *dual network flow problem*), because the constraints in (1b) are the constraints in the dual of the minimum cost flow problem. The dual network flow problem and its special cases arise in various application settings, including multi-product multistage production/inventory systems, project scheduling, machine-scheduling with precedence constraints, dial-a-ride transit problems, inverse optimization, and isotone regression. We describe some of these applications in Section 6 and provide references for other applications.

Instead of solving (1), we shall solve a problem that is equivalent to (1). We will make the following assumptions about (1):

Assumption 1: *Each* $\bar{F}_{ij}(w_{ij})$ *and* $\bar{B}_i(\mu_i)$ *is a piecewise linear convex function with integer breakpoints.* Since each function $\bar{F}_{ij}(w_{ij})$ is a convex function of the decision variable w_{ij} which is required to take integer values, we can replace $\bar{F}_{ij}(w_{ij})$ by a piecewise linear convex function $F_{ij}(w_{ij})$ with integer breakpoints (that is, we set $F_{ij}(w_{ij}) = \bar{F}_{ij}(w_{ij})$ for each integer w_{ij} in the range $[l_{ij}, u_{ij}]$ and make $F_{ij}(w_{ij})$ linear between every two consecutive integer values of w_{ij}). We perform the same transformation for each $\bar{B}_i(\mu_i)$. This assumption allows us to eliminate integrality requirements on decision variables.

Assumption 2: *There are no lower and upper bound constraints on the variables* w_{ij} *and* μ_i. We can satisfy this assumption by modifying $\bar{F}_{ij}(w_{ij})$ and $\bar{B}_i(\mu_i)$ as follows:

$$F_{ij}(w_{ij}) = \begin{cases} \bar{F}_{ij}(u_{ij}) + M(w_{ij} - u_{ij}) & \text{for } w_{ij} > u_{ij} \\ \bar{F}_{ij}(w_{ij}) & \text{for } l_{ij} \le w_{ij} \le u_{ij}\ , \\ \bar{F}_{ij}(l_{ij}) - M(w_{ij} - l_{ij}) & \text{for } w_{ij} < l_{ij} \end{cases} \tag{2a}$$

$$B_i(\mu_i) = \begin{cases} \bar{B}_i(u_i) + M(\mu_i - u_i) & \text{for } \mu_i > u_i \\ \bar{B}_i(\mu_i) & \text{for } l_i \le \mu_i \le u_i\ , \\ \bar{B}_i(l_i) - M(\mu_i - l_i) & \text{for } \mu_i < l_i \end{cases} \tag{2b}$$

where M is a sufficiently large number. We choose M such that in any solution (w, μ) that satisfies (1b) but violates (1c) or (1d) cannot be an optimal solution of (1). It is easy to see that any value of $M \ge L_f - L_b$ will suffice, where L_f denotes a feasible objective function value and L_b denotes a lower bound on the objective function value (which we can obtain by relaxing the constraints in (1b)).

Assumption 3: *Each constraint in (1b) is an equality constraint*. We will show in Section 5 that the inequality type constraints can be transformed to the equality type constraints.

We will henceforth assume that the transformation described above to satisfy Assumptions 1, 2, and 3 have been made. These transformations give us the following formulation:

$$\text{Minimize } \textstyle\sum_{(i,\, j)\in Q} F_{ij}(w_{ij}) \ + \ \sum_{i\in P} B_i(\mu_i) \tag{3a}$$

subject to

$$\mu_i - \mu_j = w_{ij} \qquad \text{for all } (i, j) \in Q. \tag{3b}$$

It is well known that (2) can be transformed into a linear programming problem by introducing a separate variable for each linear segment in the functions $F_{ij}(w_{ij})$ and $B_i(\mu_i)$. It follows from the integrality of the breakpoints (that is, those points where the slopes of the functions $F_{ij}(w_{ij})$ and $B_i(\mu_i)$ change) that there always exist an optimal solution of (3) that is integer; this justifies removing the integrality constraints on variables from the formulation.

In this paper, we develop an O(nm log n log(nU)) algorithm to solve (3). This is currently the best time bound to solve the dual network flow problem. Our algorithm follows from the following series of results:

(1) We consider the Lagrangian relaxation problem of (3) obtained by relaxing the constraints in (3b). We show that the Lagrangian multiplier problem for this relaxation (that is, the problem of obtaining the Lagrangian multipliers so that the relaxation has the maximum possible objective function value) can be transformed to a network flow problem with nonlinear costs. This result is shown in Section 2.

(2) We show that the network flow problem with the nonlinear cost described above is a convex cost network flow problem, that is, the minimum cost flow problem with piecewise linear convex functions where each linear segment has integer slopes. This result is established in Section 3.

(3) We next adapt the cost-scaling algorithm for the minimum cost flow problem, due to Goldberg and Tarjan [1987], to solve the convex cost network flow problem. Using the integrality of the slopes of the linear segments in the cost functions and some additional properties, we show that the cost-scaling algorithm can solve the convex cost network flow problem in O(nm log n log(nU)) time. This result is shown in Section 4.

We now briefly survey related research. Hochbaum and Shanthikumar [1990] showed how to solve the dual network flow problem in polynomial time. In that paper it was shown that any convex (separable) minimization over a set of constraints that form a totally unimodular matrix is solved in polynomial time in integers. The algorithm consists of O(log U) calls to a linearized version of the problem. Recently, Ahuja, Hochbaum and Orlin [1999a] demonstrated how to implement the algorithm for the linearized version of the problem by solving a minimum cut problem on an associated graph. The running times of both the preceding approaches are worse than the running times of the algorithm reported in this paper. McCormick [1998], who read earlier versions of this paper, informed us that the cancel and tighten algorithm due to Karzanov and McCormick [1997] can be modified to solve the convex cost integer dual network flow problem; in addition, the modified algorithm can be shown to achieve the running time of O(nm log n log(nU)). Hochbaum and Queyranne [1999] recently devised an algorithm for a special case of our problem, with the functions $\bar{F}_{ij}(w_{ij})$ restricted to be linear and $w_{ij} \geq 0$. That algorithm has the complexity of solving a minimum cut problem on a graph (that is, O(nm log n^2/m)) plus the complexity of finding the integer minima of the convex functions $\bar{B}_i(\mu_i)$, generally found in O(n log U). An earlier version of this paper appeared in *IPCO Proceedings* (Ahuja, Hochbaum and Orlin [1999b]).

2. Transformation to a Network Flow Problem

We use the Lagrangian relaxation technique of Rockafellar [1984] to solve the dual network flow problem. He showed that the Lagrangian multiplier problem is a minimum convex cost network flow problem. We review his approach here and include some additional lemmas. Our approach differs from his approach in a couple of aspects: we are focused on finding an optimal integral solution, and we are focused on developing formally efficient algorithms. We also include this material in this paper because our notation and basic network terminology are substantially different from that of Rockafellar.

We dualize the constraints (3b) using the vector x, obtaining the following *Lagrangian subproblem*:

$$L(x) = \min \sum_{(i,j)\in Q} F_{ij}(w_{ij}) + \sum_{i\in P} B_i(\mu_i) - \sum_{(i,j)\in Q} (w_{ij} + \mu_j - \mu_i)x_{ij} \tag{4}$$

It is easy to show that

$$\sum_{(i,j)\in Q} (\mu_j - \mu_i)\, x_{ij} = \sum_{i\in P} \mu_i\Big(\sum_{\{j:(j,i)\in Q\}} x_{ji} - \sum_{\{j:(i,j)\in Q\}} x_{ij}\Big) = \sum_{i\in P} x_{i0}\mu_i, \tag{5}$$

where $x_{i0} = \sum_{\{j:(j,i)\in Q\}} x_{ji} - \sum_{\{j:(i,j)\in Q\}} x_{ij}$ for each $i \in P$. Substituting (5) into (4) yields

$$L(x) = \min_{w,\mu} \sum_{(i,j)\in Q} \{ F_{ij}(w_{ij}) - x_{ij}\, w_{ij}\} + \sum_{i\in P}\{ B_i(\mu_i) - x_{i0}\mu_i\}, \tag{6a}$$

subject to

$$x_{i0} = \sum_{\{j:(j,i)\in Q\}} x_{ji} - \sum_{\{j:(i,j)\in Q\}} x_{ij} \qquad \text{for all } i \in P. \tag{6b}$$

We will now simplify the Lagrangian subproblem (6). We define a directed network G = (N, A) with the node set N and the arc set A. The node set N contains a node i for each element $i \in P$ and an extra node, node 0. The arc set A contains an arc (i, j) for each $(i, j) \in Q$ and an arc (i, 0) for each $i \in P$. For each arc (i, 0), $i \in P$, we let $w_{i0} = \mu_i$, $l_{i0} = l_i$, $\mu_{i0} = \mu_i$, and $F_{i0}(w_{i0}) = B_i(\mu_i)$. Observe that $|N| = n + 1$, and $|A| = m + n$. It is easy to see that in terms of these notations, the Lagrangian subproblem (6) can be restated as follows:

$$L(x) = \min_{w} \sum_{(i,\, j)\in A} \{ F_{ij}(w_{ij}) - x_{ij}w_{ij}\} \tag{7a}$$

subject to

$$\sum_{\{j:(j,i)\in A\}} x_{ji} - \sum_{\{j:(i,j)\in A\}} x_{ij} = 0 \qquad \text{for all } i \in N. \tag{7b}$$

Since each $F_{ij}(w_{ij})$ is a convex function, $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ is also a convex function of w_{ij} for a given value of x_{ij}. (When we mention of a specified value of x, we mean that we specify x_{ij} for each $(i, j) \in Q$, and the remaining x_{ij} for each $(i, j) \in A\backslash Q$ are determined using (7b).) For a given value of x, each term of $\min_{w} \sum_{(i,\, j)\in A} \{ F_{ij}(w_{ij}) - x_{ij}w_{ij}\}$ can be optimized separately. Consequently, in order to determine L(x) for a given x, we need to determine $\min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}: w_{ij} \text{ integer}\}$ for each $(i, j) \in A$.

Let $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij} : w_{ij} \text{ integer}\}$ for each $(i, j) \in A$. In terms of $H_{ij}(x_{ij})$, (7) can be restated as:

$$L(x) = \sum_{(i,\, j)\in A} H_{ij}(x_{ij}) \text{ subject to (7b).} \tag{8}$$

We now focus on solving the *Lagrangian Multiplier Problem,* which is to determine the value of x for which the Lagrangian function L(x) attains the highest objective function value. The Lagrangian multiplier problem is to determine x* such that

$$L(x^*) = \max L(x) = \max \sum_{(i,\, j)\in A} H_{ij}(x_{ij}) \text{ subject to (7b),} \tag{9a}$$

which is a network flow problem in terms of the arc flow x (which is the vector of decision variables) with nonlinear cost functions and with no upper or lower bounds on arc flows x.

The following well-known theorem establishes a connection between the Lagrangian multiplier problem and the dual network flow problem.

Theorem 1. *Let x* be an optimal solution of the Lagrangian multiplier problem (9). Then L(x*) equals the optimal objective function value of the dual network flow problem (2).*

Proof: The problem (9a) can be transformed to a linear programming problem by introducing a separate variable for each linear segment in the cost functions $H_{ij}(x_{ij})$. Then, Theorem 1 follows from a well-known result in the theory of Lagrangian relaxation (see, for example, Ahuja, Magnanti and Orlin [1993], Theorem 16.8). ♦

We will illustrate the concepts so far using a numerical example. Suppose that P = {1, 2, 3, 4} and Q = {(1, 2), (2,3), (3,4), (4,1), (2,4)}, and we wish to solve the following dual network flow problem:

$$\begin{aligned}
&\text{Minimize } \sum_{(i,j)\in Q} c_{ij}(w_{ij} - a_{ij})^2 + \sum_{i\in P} c_i(\mu_i - a_i)^3 \\
&\text{subject to} \\
&\qquad -2 \le w_{ij} \le 3 \qquad \text{for all } (i, j) \in Q, \\
&\qquad 0 \le \mu_i \le 4 \qquad \text{for all } i \in P,
\end{aligned}$$

where c_{ij}'s, a_{ij}'s, c_i's, and a_i's are specified constants. The network G = (N, A) corresponding to this dual network flow problem is shown in Figure 1. Let $c_{i0} = c_i$ and $a_{i0} = a_i$ for all $i \in P$. The Lagrangian multiplier problem is to solve (9a) where

$$H_{ij}(x_{ij}) = \begin{cases} \min\{c_{ij}(w_{ij} - a_{ij})^2 - x_{ij}w_{ij} : -2 \le w_{ij} \le 3, \text{ and } w_{ij} \text{ integer}\} & \text{for each } (i, j)\in Q \\ \min\{c_{ij}(w_{ij} - a_{ij})^3 - x_{ij}w_{ij} : 0 \le w_{ij} \le 4, \text{ and } w_{ij} \text{ integer}\} & \text{for each } (i, j)\in A \setminus Q \end{cases}$$

3. Properties of the Function $H_{ij}(x_{ij})$

In this section, we study properties of the function $H_{ij}(x_{ij})$. We shall use these properties in the next section to develop an efficient cost-scaling algorithm to solve the Lagrangian multiplier problem (9a).

We first study how to compute $H_{ij}(x_{ij})$ for a given value of x_{ij}. Recall that $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij} : w_{ij} \text{ integer}\}$. There are three cases to consider:

(i) **Case 1**: $x_{ij} > M$. It follows from (2a) that if $w_{ij} > M$, then increasing w_{ij} further increases $F_{ij}(w_{ij})$ at a rate of M per unit increase in w_{ij}. But increasing w_{ij} decreases $-x_{ij}w_{ij}$ at a rate strictly greater than M (because $x_{ij} > M$). Consequently, $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ strictly decreases if we increase w_{ij} beyond M. Hence, $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ approaches $-\infty$ as w_{ij} approaches ∞. Therefore, $H_{ij}(x_{ij}) = -\infty$ for $x_{ij} > M$.

(ii) **Case 2**: $x_{ij} < -M$. An analysis similar to that in Case 1 yields that $H_{ij}(x_{ij})$ approaches $-\infty$ as w_{ij} approaches $-\infty$. Therefore, $H_{ij}(x_{ij}) = -\infty$ for $x_{ij} < -M$.

(iii) **Case 3**: $-M \leq x_{ij} \leq M$. In this case, we will show that $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ will achieve its minimum for some w_{ij} satisfying $l_{ij} \leq w_{ij} \leq u_{ij}$. To see this, observe that for $w_{ij} > u_{ij}$, $F_{ij}(w_{ij})$ increases at a rate of M per unit increase in w_{ij}, and $-x_{ij}w_{ij}$ decreases at a rate no more than M (because $x_{ij} \leq M$); consequently, $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ increases as w_{ij} increases. A similar argument applies when $w_{ij} < l_{ij}$. Hence $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}: l_{ij} \leq w_{ij} \leq u_{ij}$ and w_{ij} integer$\}$. Since $F_{ij}(w_{ij})$ is a piecewise linear convex function, $F_{ij}(w_{ij}) - x_{ij}w_{ij}$ is also a piecewise linear convex function. Therefore, we can find $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}: l_{ij} \leq w_{ij} \leq u_{ij}$ and w_{ij} integer$\}$ by performing binary search and considering integer values of w_{ij} in the interval $[l_{ij}, u_{ij}]$, which requires O(log U) time. Thus, in this case, $H_{ij}(x_{ij})$ can be computed in O(log U) time. (Here we assume that for a specified value of w_{ij}, $F_{ij}(w_{ij})$ can be computed in O(1) time.)

Recall that the Lagrangian multiplier problem is to determine a vector x that maximizes $\Sigma_{(i,\, j)\in A}\, H_{ij}(x_{ij})$ subject to (7b). The preceding discussion implies that $H_{ij}(x_{ij}) = -\infty$ when $x_{ij} > M$ or when $x_{ij} < -M$. Thus we can exclude the values $x_{ij} > M$ and $x_{ij} < -M$ when solving the Lagrangian multiplier problem. Alternatively, the Lagrangian multiplier problem can be stated as to maximize $\Sigma_{(i,\, j)\in A}\, H_{ij}(x_{ij})$ subject to (7b) and the following additional redundant constraints:

$$-M \leq x_{ij} \leq M \quad \text{for all } (i, j) \in A, \tag{9b}$$

where $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}: w_{ij} \text{ integer}\}$. We will next show that $H_{ij}(x_{ij})$ is a piecewise linear concave function of x_{ij}. Our subsequent discussion uses the following property:

Property 1.

(a) *Let $F_{ij}(\theta)$ be a convex function of θ. Then, $F_{ij}(\theta + 1) - F_{ij}(\theta) \leq F_{ij}(\theta + 2) - F_{ij}(\theta + 1) \leq F_{ij}(\theta + 3) - F_{ij}(\theta + 2) \leq \ldots\ldots \leq F_{ij}(u_{ij}) - F_{ij}(u_{ij} - 1)$*

(b) *Let* $b_{ij}(\theta) = F_{ij}(\theta+1) - F_{ij}(\theta)$ *for* $l_{ij} \le \theta \le u_{ij} - 1$, θ *integral. Then,*
$b_{ij}(\theta) \le b_{ij}(\theta + 1) \le b_{ij}(\theta + 2) \le \dots \le b_{ij}(u_{ij} - 1)$.

Property 1(a) directly follows from the convexity of the function $F_{ij}(\theta)$, and Property 1(b) is equivalent to Property 1(a). Since for a given x_{ij}, $F_{ij}(w_{ij}) - w_{ij}x_{ij}$ is a piecewise linear convex function of w_{ij}, $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}\text{: } w_{ij} \text{ integer}\} = \min\{F_{ij}(w_{ij}) - x_{ij}w_{ij}\text{: } l_{ij} \le w_{ij} \le u_{ij} \text{ and } w_{ij} \text{ integer}\} = \min\{F_{ij}(l_{ij}) - x_{ij}l_{ij}, F_{ij}(l_{ij}+1) - x_{ij}(l_{ij}+1), F_{ij}(l_{ij}+2) - x_{ij}(l_{ij}+2), \dots , F_{ij}(u_{ij}) - x_{ij}u_{ij}\}$. Observe that the function $H_{ij}(x_{ij})$ is the lower envelope of the lines $F_{ij}(l_{ij}) - l_{ij}x_{ij}$, $F_{ij}(l_{ij}+1) - (l_{ij}+1)x_{ij}$, $F_{ij}(l_{ij}+2) - (l_{ij}+2)x_{ij}, \dots , F_{ij}(u_{ij}) - u_{ij}x_{ij}$ and hence is a piecewise linear concave function.

Theorem 2. *The function* $H_{ij}(x_{ij}) = \min\{F_{ij}(w_{ij}) - w_{ij}x_{ij} : w_{ij} \text{ integer}\}$ *is a piecewise linear concave function of* x_{ij}, *and is described in the following manner:*

$$H_{ij}(x_{ij}) = \begin{cases} F_{ij}(l_{ij}) - l_{ij}x_{ij} & \text{if } -M \le x_{ij} \le b_{ij}(l_{ij}) \\ F_{ij}(l_{ij}+1) - (l_{ij}+1)x_{ij} & \text{if } b_{ij}(l_{ij}) \le x_{ij} \le b_{ij}(l_{ij}+1) \\ : & \\ F_{ij}(q) - qx_{ij} & \text{if } b_{ij}(q-1) \le x_{ij} \le b_{ij}(q) \\ : & \\ F_{ij}(u_{ij}) - u_{ij}x_{ij} & \text{if } b_{ij}(u_{ij}-1) \le x_{ij} \le M \end{cases} \tag{10}$$

Proof. Consider the lines $F_{ij}(\theta) - \theta x_{ij}$ as θ varies from l_{ij} to u_{ij}. The line $F_{ij}(l_{ij}) - l_{ij}x_{ij}$ intersects with the line $F_{ij}(l_{ij}+1) - (l_{ij}+1)x_{ij}$ at $x_{ij} = F_{ij}(l_{ij}+1) - F_{ij}(l_{ij}) = b_{ij}(l_{ij})$. Similarly, the line $F_{ij}(l_{ij}+1) - (l_{ij}+1)x_{ij}$ intersects with the line $F_{ij}(l_{ij}+2) - (l_{ij}+2)x_{ij}$ at $x_{ij} = b_{ij}(l_{ij}+1)$. It follows from Property 1(b) that $b_{ij}(l_{ij}) \le b_{ij}(l_{ij}+1)$. In general, the line $F_{ij}(\theta) - \theta x_{ij}$ intersects with the line $F_{ij}(\theta+1) - (\theta+1)x_{ij}$ at $x_{ij} = b_{ij}(\theta)$ for each $\theta = l_{ij}, l_{ij}+1, l_{ij}+2, \dots , u_{ij}-1$. This together with the fact that $b_{ij}(l_{ij}) \le b_{ij}(l_{ij} + 1) \le b_{ij}(l_{ij} + 2) \le \dots \le b_{ij}(u_{ij})$ establishes the theorem.

♦

It follows from Theorem 2 that the line $F_{ij}(\theta) - \theta x_{ij}$ for a given θ, $l_{ij} \le \theta \le u_{ij}$, is represented in $H_{ij}(x_{ij})$ for $x_{ij} \in [b_{ij}(\theta-1), b_{ij}(\theta)]$ (where $b_{ij}(l_{ij}-1) = -M$ and $b_{ij}(u_{ij}) = M$). If $b_{ij}(l_{ij}-1) < b_{ij}(l_{ij}) < b_{ij}(l_{ij}+1) < b_{ij}(l_{ij}+2) < \dots < b_{ij}(u_{ij})$, then each line $F_{ij}(\theta) - \theta x_{ij}$ contributes a linear segment of positive length to $H_{ij}(x_{ij})$; in this case, slopes of the linear segments in the function $H_{ij}(x_{ij})$ take all the values from $-l_{ij}$ to $-u_{ij}$ as x_{ij} varies from -M to M. However, if $b_{ij}(\theta-1) = b_{ij}(\theta)$ for some θ, then $F_{ij}(\theta) - \theta x_{ij}$ will contribute just a point (or a line segment of zero length) to $H_{ij}(x_{ij})$. In this case, slopes of the linear segments in the function $H_{ij}(x_{ij})$ will not take all the values from $-l_{ij}$ to $-u_{ij}$ as x_{ij} varies from -M to M; instead at some breakpoints, the slope will change by more than one unit.

We next define the right and left slopes of $H_{ij}(x_{ij})$. For the function $H_{ij}(x_{ij})$, we define its *right slope* at point x_{ij} as $(H_{ij}(x_{ij}+\delta) – H_{ij}(x_{ij}))/\delta$, and its *left slope* at point x_{ij} as $(H_{ij}(x_{ij}) – H_{ij}(x_{ij}-\delta)/\delta$, where δ is a sufficiently small number. In other words, the right slope of $H_{ij}(x_{ij})$ at point x_{ij} equals the slope of the linear segment in $H_{ij}(x_{ij})$ on the right side of the point x_{ij} and the left slope equals the negative of the slope of the linear segment on the left side of the point x_{ij}. It follows from Theorem 2 and our preceding discussion that at $x_{ij} = b_{ij}(\theta)$, the right slope of the function $H_{ij}(x_{ij})$ is at most $–(\theta+1)$ and its left slope is at least $-\theta$.

We illustrate the preceding definitions using a numerical example. In our example described in Section 2, consider an arc $(i, j) \in Q$ with $F_{ij}(w_{ij}) = c_{ij}(w_{ij} – a_{ij})^2$ and $–2 \le w_{ij} \le 3$. Suppose that $c_{ij} = 1$ and $a_{ij} = 0$. For this arc,

$$H_{ij}(x_{ij}) = \min\{4+2x_{ij}, 1+2x_{ij}, 0, 1-x_{ij}, 4-2x_{ij}, 9-3x_{ij}\}.$$

We show in Figure 2 these lines. The lower envelope of these lines defines $H_{ij}(x_{ij})$. The expression for $H_{ij}(x_{ij})$ is given below:

$$H_{ij}(x_{ij}) = \begin{cases} 4 + 2x_{ij} & \text{if } -M \le x_{ij} \le -3 \\ 1 + x_{ij} & \text{if } -3 \le x_{ij} \le -1 \\ 0 & \text{if } -1 \le x_{ij} \le 1 \\ 1 - x_{ij} & \text{if } 1 \le x_{ij} \le 3 \\ 4 - 2x_{ij} & \text{if } 3 \le x_{ij} \le 5 \\ 9 - 3x_{ij} & \text{if } 5 \le x_{ij} \le M \end{cases}$$

We have so far transformed the Lagrangian multiplier problem to maximizing a concave cost flow problem. We can alternatively restate this problem as

$$\text{Minimize } \sum_{(i,\, j)\in A} C_{ij}(x_{ij}) \tag{11a}$$

subject to

$$\sum_{\{j:(j,i)\in A} x_{ji} - \sum_{\{j:(i,j)\in A} x_{ij} = 0 \qquad \text{for all } i \in N, \tag{11b}$$

$$-M \le x_{ij} \le M \qquad \text{for all } (i, j) \in A, \tag{11c}$$

where $C_{ij}(x_{ij}) = -H_{ij}(x_{ij})$. The slopes of the linear segments in $C_{ij}(x_{ij})$ are the negatives of the corresponding slopes in $H_{ij}(x_{ij})$. Since $H_{ij}(x_{ij})$ is a piecewise linear concave function, $C_{ij}(x_{ij})$ is a piecewise linear convex function. The proof of the next lemma follows from this correspondence and the preceding discussion:

Lemma 1. *If* $x_{ij} = b_{ij}(\theta)$ *for some integer* $\theta \in [l_{ij}, u_{ij}-1]$*, then the right slope of* $C_{ij}(x_{ij})$ *is at least* $\theta+1$ *and its left slope is at most* θ*. In addition, the right slope of* $C_{ij}(x_{ij})$ *at* $x_{ij} = -M$ *is* l_{ij}*, and the left slope of* $C_{ij}(x_{ij})$ *at* $x_{ij} = M$ *is* u_{ij}.

4. Cost-Scaling Algorithm

The problem (11) is a convex cost flow problem in which the cost associated with the flow on arc (i, j) is a piecewise linear convex function containing at most U+1 linear segments. We can transform (11) into a minimum cost flow problem in an expanded network G′ = (N′, A′), and solve it using any minimum cost flow algorithm. In this transformation, for each arc (i, j) ∈ A, we introduce $(u_{ij}-l_{ij}+1)$ arcs in G′- one corresponding to each linear segment, and the costs of these arcs are: l_{ij}, $l_{ij}+1$, $l_{ij}+2$, … , u_{ij}, and the capacities of these arcs, respectively, are $b_{ij}(l_{ij})+M$, $b_{ij}(l_{ij}+1)-b_{ij}(l_{ij})$, $b_{ij}(l_{ij}+2)-b_{ij}(l_{ij}+1)$, …. , $M-b_{ij}(u_{ij}-1)$. Notice that we introduce U+1 arcs in G′ for each arc (i, j) in G and the cost of each arc is also bounded by U. It is well known that solving the convex cost flow problem in G is equivalent to solving the minimum cost flow problem in G′. Hence we can use any minimum cost flow algorithm to solve the minimum cost flow problem in G′. However, since the number of arcs in the network G are O(nU), the minimum cost flow algorithms would not in general run in polynomial time. In addition, some minimum cost flow algorithms, such as capacity scaling algorithms, require that the arc capacities are integer. In the above transformation, arc capacities can be non-integer and capacity scaling algorithms may not be applied.

We can, however, use the cost-scaling algorithm due to Goldberg and Tarjan [1987] to solve the minimum cost flow problem in G′. The correctness of the cost-scaling algorithm relies on the fact that all arc costs must be integer. Observe that the minimum cost flow problem in G′ has integer costs. Consequently, the cost-scaling algorithm will correctly solve the minimum cost flow problem in G′. However, the running time of the cost-scaling algorithm will not be polynomial. We now describe a modification of the cost-scaling algorithm which can solve the convex cost flow problem in (11) in about the same time as needed by the cost-scaling algorithm to solve the minimum cost flow problem. We will subsequently refer to the modified algorithm as the *convex cost-scaling algorithm* and the cost-scaling algorithm for the minimum cost flow problem as the *linear cost-scaling algorithm*.

Our subsequent discussion requires an understanding of the linear cost-scaling algorithm for the minimum cost flow problem. We refer the readers to the paper by Goldberg and Tarjan [1987] or the book of Ahuja, Magnanti and Orlin [1993] for a description of this algorithm. Goldberg and Tarjan had observed that their linear cost-scaling algorithm could be extended to treat convex cost flows at each scaling phase; however, the algorithm is not guaranteed to find an optimal solution of the convex cost flow problem in a polynomially bounded number of scaling phases (Tarjan [1998] communicated this to one of the co-authors of this paper). We will show that due to the special structure of (11), the modified cost-scaling algorithm obtains an optimal solution of the convex cost flow problem. In addition, we show that the running time of the modified cost-scaling algorithm to solve (11) is almost the same time as is needed to solve the minimum cost flow problem.

We now briefly describe the linear cost-scaling algorithm here and point out the changes we need to make in order to apply it for the convex cost case. We will use the notation given in Ahuja, Magnanti and Orlin [1993]. We consider the minimum cost flow problem with c_{ij}'s as arc costs, u_{ij}'s as arc capacities, and zero lower bounds on arc costs. The cost-scaling algorithm maintains a pseudoflow at each step. A *pseudoflow* x is any function $x : A \rightarrow R$ that satisfies upper and lower bounds on arc flows but may violate the flow balance constraints at nodes. For any pseudoflow x, we define the *imbalance* of node i as $e(i) = \sum_{\{j:(j,i)\in A\}} x_{ji} - \sum_{\{j:(i,j)\in A\}} x_{ij}$ for all $i \in N$. If $e(i) > 0$ for some node i, we refer to node i as an *excess node* and refer to e(i) as the *excess* of node i. We refer to a pseudoflow x with $e(i) = 0$ for all $i \in N$ as a *flow*.

We henceforth assume for notational convenience that for any pair of nodes i and j, either $(i, j) \in A$ or $(j, i) \in A$, but not both. We can easily satisfy this assumption by performing a simple transformation, but the assumption is not needed in any case.

The cost-scaling algorithm also maintains a value $\pi(i)$ for each node $i \in N$. We refer to the vector π as a vector of *node potentials*. The cost-scaling algorithm proceeds by constructing and manipulating the residual network G(x) defined as follows with respect to a pseudoflow x. For each arc $(i, j) \in A$, the residual network G(x) contains two arcs (i, j) and (j, i). The arc (i, j) has cost c_{ij} and residual capacity $r_{ij} = u_{ij} - x_{ij}$, and the arc (j, i) has cost $c_{ji} = -c_{ij}$ and residual capacity $r_{ji} = x_{ij}$. The residual network consists *only* of arcs with positive residual capacity.

For a given residual network G(x) and a set of node potentials π, we define the *reduced cost* of an arc (i, j) as $c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j)$. For a given value of ε, we call an arc (i, j) *admissible* if $-\varepsilon \leq c^{\pi}_{ij} < 0$. A flow or a pseudoflow x is said to be *ε-optimal* for some $\varepsilon \geq 0$ if for some node potentials π, the pair (x, π) satisfies the following *ε-optimality conditions*: $c^{\pi}_{ij} \geq -\varepsilon$ for every arc (i, j) in G(x). The cost-scaling algorithm treats ε as a parameter and iteratively obtains ε-optimal flows for successively smaller values of ε. Initially, $\varepsilon = \max \{c_{ij} : (i, j) \in A\} = U$ and any feasible flow is ε-optimal. The algorithm then performs cost-scaling phases by repeatedly applying an improve-approximation procedure that transforms an ε-optimal flow into a $\frac{\varepsilon}{2}$-optimal flow. After O(log(nU)) cost-scaling phases, $\varepsilon < 1/n$ and the algorithm terminates with an optimal flow.

The basic operation in the improve-approximation procedure is to select an excess node i, and perform *pushes* on admissible arcs emanating from it. The amount pushed on an admissible arc (i, j) is $\min\{r_{ij}, e(i)\}$. If $e(i) < r_{ij}$, then the push is a *nonsaturating* push; otherwise it is a *saturating* push. When node i has no admissible arc emanating from it, then it increases the potential of node by $\varepsilon/2$; this step is called a *relabel* step. The improve-approximation procedure terminates when there is no excess node (that is, the current pseudoflow is a flow). Hence the linear cost-scaling algorithm performs three major operations per scaling phase: there are O(nm) saturating pushes, $O(n^2m)$ nonsaturating pushes, and $O(n^2)$ relabels which take O(nm) time in total during each scaling phase.

We now describe how to modify the cost-scaling algorithm for the convex cost case. For a given arc flow x′, we construct the residual network G(x′) in the usual manner. For the flow x′, the residual capacity of (i, j) is $r_{ij} = M - x'_{ij}$, and the residual capacity of arc (j, i) is $x'_{ij} + M$. The residual network G(x′) consists of arcs with positive residual capacity only. We refer to an arc (i, j) in G(x′) as a *forward arc* if (i, j) ∈ A, and refer to it as a *backward arc* if (j, i) ∈ A. We denote by A(i) the set of arcs in G(x′) emanating from node i

For a given flow x′, we set the cost $c_{ij}(x')$ of a forward arc (i, j) as the right slope of $H_{ij}(\cdot)$ at x'_{ij}. We define the cost $c_{ji}(x')$ of a backward arc (j, i) as the negative of the left slope of $H_{ij}(\cdot)$ at x'_{ij}. Note that if H were a linear function cx, then the cost of arc (i, j) would be c_{ij}, and the cost of the backward arc (j, i) would be $-c_{ij}$, which is the same as the costs of the arcs in G(x′) in the linear cost-scaling algorithm. We define the *reduced cost* of a forward arc (i, j) in G(x′) as $c^{\pi}_{ij}(x') = c_{ij}(x') - \pi(i) + \pi(j)$, and the reduced cost of a backward arc (j, i) as $c^{\pi}_{ji}(x') = c_{ji}(x') - \pi(j) + \pi(i)$.

We say that an arc (i, j) is *violating* with respect to pseudo-flow x′ and vector π of potentials if arc (i, j) ∈ G(x′) and $c^{\pi}_{ij}(x') < 0$; otherwise we say that arc (i, j) is *nonviolating*. If an arc (i, j) is violating, we let q_{ij} denote an amount of flow that we need to send on arc (i, j) so that both the arcs (i, j) and (j, i) are nonviolating after the push. Observe that if we send q_{ij} amount of flow on a forward arc (i, j) in G(x′) then the flow on the arc (i, j) increases by q_{ij} units, and if we send q_{ij} amount of flow on a backward arc (j, i) then the flow on the arc (i, j) decreases by q_{ij} units.

For a violating forward arc (i, j) in G(x′), we define q_{ij} as follows:

$$q_{ij} = \begin{cases} M - x'_{ij} & \text{if } \pi(i) - \pi(j) \geq u_{ij} \\ b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor) - x'_{ij} & \text{if } \pi(i) - \pi(j) < u_{ij} \end{cases}, \tag{12a}$$

and for a violating backward arc (j, i) in G(x′), we define q_{ji} as follows:

$$q_{ij} = \begin{cases} M + x'_{ij} & \text{if } \pi(i) - \pi(j) \leq l_{ij} \\ x'_{ij} - b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor) & \text{if } \pi(i) - \pi(j) > l_{ij} \end{cases}. \tag{12b}$$

Lemma 2. *If arc (i, j) in G(x′) is a violating arc (either forward or backward), and if* q_{ij} *units of flow are sent on arc (i, j), then subsequently neither the arc (i, j) nor the arc (j, i) is violating.*

Proof. Let x'' denote the flow after the push. There are four cases to consider.

Case 1: *Arc (i, j) is a forward arc and* $\pi(i) - \pi(j) \geq u_{ij}$. In this case, as defined by (12), $q_{ij} = M - x'_{ij}$, and the flow x''_{ij} after the push is $x''_{ij} = M$, and thus arc $(i, j) \notin G(x'')$. Moreover, the left slope of $C_{ij}(M)$ (by Lemma 1) equals u_{ij} (or, $c_{ji}(x'') = -u_{ij}$). Thus, $c^{\pi}_{ji}(x'') = c_{ji}(x'') + \pi(i) - \pi(j) = -u_{ij} + \pi(i) - \pi(j) \geq 0$.

Case 2: *Arc (i, j) is a forward arc and* $\pi(i) - \pi(j) < u_{ij}$. In this case, $q_{ij} = b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor) - x'_{ij}$, and so after the push, $x''_{ij} = b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor)$. By Lemma 1, the right slope of $C_{ij}(x''_{ij})$ is at least $\lfloor \pi(i) - \pi(j) \rfloor + 1$ (or, $c_{ij}(x'') \geq \lfloor \pi(i) - \pi(j) \rfloor + 1$) and, therefore,

$$c^{\pi}_{ij}(x'') = c_{ij}(x'') - \pi(i) + \pi(j) \geq (\lfloor \pi(i) - \pi(j) \rfloor + 1) - (\pi(i) - \pi(j)) \geq 0.$$

Also by Lemma 1, the left slope of $C_{ij}(x''_{ij})$ is at most $\lfloor \pi(i) - \pi(j) \rfloor$ (or, $c_{ji}(x'') \geq -\lfloor \pi(i) - \pi(j) \rfloor$) and, therefore,

$$c^{\pi}_{ji}(x'') = c_{ji}(x'') + \pi(i) - \pi(j) \geq -\lfloor \pi(i) - \pi(j) \rfloor + (\pi(i) - \pi(j)) \geq 0.$$

Case 3: *Arc (j, i) is a backward arc, and* $\pi(i) - \pi(j) \leq l_{ij}$. In this case, $x''_{ij} = -M$, and after the push $(j, i) \notin G(x'')$. Moreover, the right slope of $C_{ij}(-M)$ by Lemma 1 equals l_{ij}, and thus $c^{\pi}_{ij}(x'') = c_{ij} - \pi(i) + \pi(j) = l_{ij} - \pi(i) + \pi(j) \geq 0$.

Case 4: *Arc (j, i) is a backward arc, and* $\pi(i) - \pi(j) > l_{ij}$. In this case, $q_{ij} = x'_{ij} - b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor)$, and after the push, the amount of flow in arc (i, j) is $b_{ij}(\lfloor \pi(i) - \pi(j) \rfloor)$, and so the argument given in Case 2 applies.

We also need to prove that $q_{ij} > 0$. We will prove this result for the case when arc (i, j) is a forward arc; proof for the case when arc (i, j) is a backward arc is similar. Observe that $c^{\pi}_{ij}(x') < 0$ (because arc (i, j) was a violating arc before the push) and $c^{\pi}_{ij}(x'') \geq 0$. By the convexity of $C_{ij}(.)$, it follows that $x'_{ij} < x''_{ij}$ and so $q_{ij} > 0$. This completes the proof of the lemma. ♦

Our convex cost-scaling algorithm is the same as the linear cost-scaling algorithm. It proceeds by identifying excess nodes and pushing flows on arcs emanating from these nodes; when there is no such arc on which flow can be pushed it relabels the node. However, it differs on two counts: (i) instead of sending flows on admissible arcs, it sends flows on violating arcs, and (ii) instead of sending $\delta = \min\{e(i), r_{ij}\}$ units of flow, it sends $\delta = \min\{e(i), q_{ij}\}$ units of flow. In the convex cost-scaling algorithm, we refer to the push on arc (i, j) as *emptying* if $\delta = e(i)$ and *nonemptying* if $\delta < e(i)$ (or, alternatively, $\delta = q_{ij}$). We show next in Lemma 2 that a nonemptying push on a violated arc (i, j) makes both the arcs (i, j) and (j, i) nonviolating. (A nonemptying push is similar to the saturating push in the linear cost-scaling algorithm which also makes both the arcs (i, j) and (j, i) inadmissible.) We now bound the running time of the convex cost-scaling algorithm. We accomplish it by bounding the number of relabel operations, the number of nonemptying pushes, and the number of emptying pushes performed in the improve-approximation procedure.

Lemma 3. *During an execution of the improve-approximation procedure, any node potential increases O(n) times.*

Proof. Instead of proving this result from scratch, we will use a similar result for the minimum cost flow problem. As described earlier, we can transform the convex cost flow problem (11) in the network $G = (N, A)$ into a minimum cost flow problem in the network $G' = (N', A')$. Goldberg and Tarjan [1987] established (see, for example, Lemma 10.4 in Ahuja, Magnanti and Orlin [1993]) that during an execution of the improve-approximation procedure no node potential increases more than 3n times. Observe that the bound depends on the number of nodes and not on the number of arcs. Consequently, in G' any node potential will increase O(n) times during an execution of the improve-approximation procedure. ♦

Lemma 4. *The improve-approximation procedure performs O(nm) nonemptying pushes.*

Proof. We will use the potential function argument on F to bound the number of nonemptying pushes. Let F denote the number of violating arcs in the residual network $G(x')$. A nonemptying push on arc (i, j) makes arc (i, j) nonviolating after the push and does not make (j, i) violating, and so it reduces F by one. An emptying push either decreases F by one or keeps it constant. Each distance increase of node i may create as many as $|A(i)|$ new violating arcs, and hence may increase F by at most $|A(i)|$. The total increase in F over all iterations is $O(n \sum_{i \in N} |A(i)|) = O(nm)$. It follows that the number of nonemptying pushes is O(nm). ♦

Lemma 5. *The improve-approximation procedure performs $O(n^2m)$ emptying pushes.*

Proof. Let E denote the set of excess nodes and for an excess node $i \in E$, let g(i) denote the number of nodes in the residual network that are reachable from node i via a directed path of violating admissible arcs. Let $\Phi = \sum_{i \in E} g(i)$. Each emptying push decreases Φ by at least 1. Each nonemptying push increases Φ by at most n. Each distance increase of node i increases Φ by at most n. A standard potential function argument (see, for example, Lemma 10.7 in Ahuja, Magnanti and Orlin [1993]) shows that the number of emptying pushes is $O(n^2m)$. ♦

The preceding lemmas establish a bound of $O(n^2m)$ on the running time of an execution of the improve-approximation procedure. After O(log(nU)) executions of improve-approximation procedure, the algorithm obtains an ε-optimal flow with $\varepsilon < 1/n$. It is easy to see that this flow is an optimal flow for the convex cost flow problem (11). This result follows from the fact that the convex cost flow problem (11) can be transformed to a minimum cost flow problem with integer arc costs and any ε-optimal flow with $\varepsilon < 1/n$ is an optimal flow for the minimum cost flow problem. We have thus shown that the convex cost-scaling algorithm solves (11) in $O(n^2m \log(nU))$ time. Using the dynamic tree data structure of Sleator and Tarjan [1983], this algorithm can be implemented in $O(nm \log n \log(nU))$ time. We conjecture that the running time can be improved further to $O(nm \log(n^2/m) \log(nU))$ using dynamic trees data structures and examining excess nodes in a specific order in the improve-approximation procedure.

The convex cost-scaling algorithm upon termination gives an optimal flow x^* and the optimal node potentials π^*. Both the solutions x^* and π^* may be non-integer. Since the objective function in the convex cost flow problem (11) is piecewise linear, it follows that there always exists an integer optimal node potentials π. To determine those, we construct $G(x^*)$ and solve a shortest path problem to determine shortest path distance $d(i)$ from node 0 to every other node $i \in N$. Since all arc costs in $G(x^*)$ are integer, each $d(i)$ is also integer. Then $\pi(i) = -d(i)$ for each $i \in N$ gives an integer optimal set of node potentials for the problem (11). Now recall that $C_{ij}(x_{ij}) = -H_{ij}(x_{ij})$ for each $(i, j) \in A$. This implies that $\mu(i) = -\pi(i) = d(i)$ for each $i \in N$ gives optimal dual variables for (9) and these $\mu(i)$ together $w_{ij} = \mu(i) - \mu(j)$ for each $(i, j) \in A$ give an optimal solution of the dual network flow problem (3). We summarize our discussion with the following theorem:

Theorem 3. *The convex cost-scaling algorithm correctly solves the dual network flow problem in $O(nm \log n \log(nU))$ time.*

5. Generalizations of the Dual Network Flow Problem

In our formulation of the dual network flow problem we have assumed that the constraints $\mu_i - \mu_j = w_{ij}$ are in the equality form. We will show in this section that the constraints of the forms $\mu_i - \mu_j \le w_{ij}$ and $\mu_i - \mu_j \ge w_{ij}$ can be transformed to the equality form; hence, there is no loss of generality by restricting the constraints to the equality form. We will first consider the case when constraints are of the form $\mu_i - \mu_j \le w_{ij}$.

Suppose that we wish to solve the following problem:

$$\text{Minimize } \textstyle\sum_{(i,\, j)\in Q} F_{ij}(w_{ij}) + \sum_{i\in P} B_i(\mu_i) \qquad (13a)$$

subject to

$$\mu_i - \mu_j \le w_{ij} \qquad \text{for all } (i, j) \in Q, \qquad (13b)$$

Let w^*_{ij} denote the value of w_{ij} for which $F_{ij}(w_{ij})$ is minimum. In case there are multiple values for which $F_{ij}(w_{ij})$ is minimum, choose the minimum such value. Let us define the function $E_{ij}(w_{ij})$ in the following manner:

$$E_{ij}(w_{ij}) = \begin{cases} F_{ij}(w^*_{ij}) & \text{if } w_{ij} \le w^*_{ij} \\ F_{ij}(w_{ij}) & \text{if } w_{ij} > w^*_{ij} \end{cases} \tag{14}$$

Now consider the following problem:

$$\text{Minimize } \sum\nolimits_{(i,\, j) \in Q} E_{ij}(w_{ij}) + \sum\nolimits_{i \in P} B_i(\mu_i) \tag{15a}$$

subject to

$$\mu_i - \mu_j = w_{ij} \qquad \text{for all } (i, j) \in Q, \tag{15b}$$

The following lemma establishes a relationship between optimal solutions of (13) and (15).

Lemma 6. *For every optimal solution ($\bar{w}$, $\bar{\mu}$) of (13), there is an optimal solution ($\hat{w}$, $\bar{\mu}$) of (15) of the same cost, and the converse also holds.*

Proof. Consider an optimal solution ($\bar{w}$, $\bar{\mu}$) of (13). We will show how to construct an optimal solution ($\hat{w}$, $\bar{\mu}$) of (15) with the same cost. Consider some $(i, j) \in Q$. There are two cases to consider:

Case 1: $\bar{\mu}_i - \bar{\mu}_j > w^*_{ij}$. It follows from (13b) and the convexity of the function $F_{ij}(w_{ij})$ that $\bar{w}_{ij} = \bar{\mu}_i - \bar{\mu}_j$. In this case, we set $\hat{w}_{ij} = \bar{w}_{ij}$. It follows from (14) that $F_{ij}(\bar{w}_{ij}) = E_{ij}(\hat{w}_{ij})$.

Case 2: $\bar{\mu}_i - \bar{\mu}_j \le w^*_{ij}$. It follows from (13b) and the convexity of the function $F_{ij}(w_{ij})$ that $\bar{w}_{ij} = w^*_{ij}$. In this case, we set $\hat{w}_{ij} = \bar{\mu}_i - \bar{\mu}_j$. It follows from (14) that $F_{ij}(\bar{w}_{ij}) = E_{ij}(\hat{w}_{ij})$.

We have thus shown that given an optimal solution of (13) how can we construct a solution of the same cost for (15). Similarly, it can be shown that if ($\hat{w}$, $\hat{\mu}$) is an optimal solution of (15), then the solution ($\bar{w}$, $\hat{\mu}$) constructed in the following manner is an optimal solution of (13): $\bar{w}_{ij} = \max\{w^*_{ij}, \hat{w}_{ij}\}$. This concludes the proof of the lemma. ♦

We have shown above how can we handle constraints of the form $\mu_i - \mu_j \le w_{ij}$ by the function $F_{ij}(w_{ij})$. Similarly, we can handle constraints of the form $\mu_i - \mu_j \ge w_{ij}$ by replacing the function $F_{ij}(w_{ij})$

by the function $E_{ij}(x_{ij})$ defined as follows: $E_{ij}(w_{ij}) = F_{ij}(w_{ij})$ if $w_{ij} \leq w_{ij}^*$, and $E_{ij}(w_{ij}) = F_{ij}(w_{ij}^*)$ if $w_{ij} > w_{ij}^*$. The proof for this case is similar to that of Lemma 6.

6. Application of the Dual Network Flow Problem

The dual network flow problem and its special cases arise in many application settings. Roundy [1986] formulates a lot-sizing problem in a multi-product, multi-stage, production/inventory system as a dual network flow problem. Boros and Shamir [1991] describe an application of the dual network flow problem in solving a quadratic cost machine scheduling problem. Several multi-facility location problems have constraints of the form (1b) – (1e) (see for example, Ahuja Magnanti and Orlin [1993, Chapter 19]). The convex cost version of these location problems will be dual network flow problems. We describe next in detail five applications of dual network flow problems, some of which we have encountered in our previous research.

Application 1. Dial-A-Ride Transit Problem

Dial-a-ride transit problems are vehicle routing problems with time windows and have been extensively studied in the literature. (See, for example, Desrosiers et al. [1995].) In this problem, customers call a dial-a-ride agency sufficiently in advance (say, one day before) requesting to be carried from specific origins to specific destinations during specified times. The agency dispatches a vehicle to meet several such demands and customers are pooled to reduce the operational costs. A vehicle schedule typically consists of picking up and dropping off of some customers in a specific sequence, and at any point of time several customers can be on board the vehicle. Researchers have developed exact as well as heuristic algorithms for dial-a-ride transit problems. Since exact algorithms can solve only small sized problems, heuristic algorithms have been more extensively studied. Some heuristic algorithms deal separately with the following two functions: routing and scheduling. The routing part determines the route of each vehicle - the order in which specific customers assigned to a vehicle will be picked up and delivered. The scheduling part assigns a time schedule to the route - the times at which the customers will be picked up and delivered. We will show that determining the optimal schedule for a given route can be formulated as the convex cost dual network flow problem.

In this scheduling problem we are given a sequence of stops (on an increasing time scale) 1-2-3-... -n, where each stop denotes a pickup or a delivery point. Let **S** denote the set of pickup stops, and for each pickup stop $i \in \mathbf{S}$ let d(i) denote the corresponding delivery stop. We assume that the vehicle takes t_j time to go from stop j to stop j+1. Each stop j has a pickup/delivery time window $[l_j, u_j]$ and also has a *desired* pickup/delivery time a_j. We penalize the deviations from the desired pickup and delivery times using a convex function. If the vehicle visits stop j at time μ_j, the penalty is given by a convex function $B_j(\mu_j - a_j)$. We also allow the vehicle to wait between stops, but penalize these waiting times. If w_i denotes the waiting time w_i between stop i and stop i+1, then $w_i = \mu_{i+1} - \mu_i - t_i$. The penalty associated with this waiting time is given by a convex function $F_i(w_i)$. Let τ_i denote the minimum ride time from a pickup stop i to its delivery stop d(i). However, due to the waiting times between stops, the actual ride time, given by $\mu_{d(i)} - \mu_i$, may be larger than τ_i; giving us the excess ride time $e_i = \mu_{d(i)} - \mu_i - \tau_i$. We penalize the excess ride time e_i using the convex function $E_i(e_i)$. Then the scheduling problem is to determine the vehicle schedule so that the total penalty incurred due to the deviations from the desired pickup and delivery time, waiting times, and the excess ride times is minimum. This problem can be formulated as the following optimization problem:

$$\text{Minimize } \sum_{j=1}^{n} [F_i(w_i) + B_i(\mu_i - a_i)] + \sum_{i \in P} E_i(e_i) \tag{16a}$$

subject to

$$w_i = \mu_{i+1} - \mu_i - t_i \qquad \text{for each } i = 1, 2, \ldots, n, \tag{16b}$$

$$e_i = \mu_{d(i)} - \mu_i - \tau_i \qquad \text{for each } i \in \mathbf{S}, \tag{16c}$$

$$0 \le w_i \le \overline{w} \qquad \text{for each } i = 1, 2, \ldots, n. \tag{16d}$$

$$0 \le e_i \le \overline{e}_i \qquad \text{for each } i \in \mathbf{S}, \tag{16e}$$

$$\mu_i, w_i, \text{ and } e_i \text{ are all integer,} \tag{16f}$$

where $\overline{w}$ specifies an upper bound on any single waiting time, and $\overline{e}_i$ specifies an upper bound on e_i. This problem can easily be formulated as the dual network flow problem by defining the set P and Q approximately.

Application 2: Inverse Spanning Tree Problem with Convex Costs

Consider an undirected network G = (N, A) with the node set N and the arc set A. Let n = |N| and m = |A|. We assume that N = {1, 2, ... , n} and A = $\{a_1, a_2, \ldots, a_m\}$. Let c_j denote the cost of the arc a_j. In the inverse spanning tree problem we are given a spanning tree T^0 of G which may or may not be a minimum spanning tree of G and we wish to perturb the arc cost vector c to d so that T^0 becomes a minimum spanning tree with d as the cost vector and $\sum_{j=1}^{n} F_j(d_j - c_j)$ is minimum, where each $F_j(d_j - c_j)$ is a convex function of d_j. Sokkalingam, Ahuja and Orlin [2000], and Ahuja and Orlin [1999] have studied special cases of the inverse spanning tree problem with cost functions as $\sum_{j=1}^{n} |d_j - c_j|$ and $\max\{|d_j - c_j| : 1 \le j \le m\}$.

We assume without any loss of generality that $T^0 = \{a_1, a_2, \ldots, a_{n-1}\}$. We refer to the arcs in T^0 as *tree arcs* and the arcs not in T^0 as *nontree arcs*. In the given spanning tree T^0, there is a unique path between any two nodes; we denote by $W[a_j]$ the set of tree arcs contained between the two endpoints of the arc a_j. It is well known (see, for example, Ahuja, Magnanti and Orlin [1993]) that T^0 is a minimum spanning tree with respect to the arc cost vector d if and only if

$$d_i \leq d_j \text{ for each } a_i \in W[a_j] \text{ and for each } j = n, n+1, \ldots, m. \tag{17}$$

We can convert the inequalities in (17) into equations by introducing nonnegative slack variables w_{ij}'s. Let $P = \{1, 2, \ldots, m\}$ and $Q = \{(i, j) : a_i \in T^0 \text{ and } a_j \in W[a_i]\}$. Then, in this notation, the inverse spanning tree problem can be formulated as the following optimization problem:

$$\text{Minimize } \sum_{i \in P} F_i(d_i - c_i) \tag{18a}$$

subject to

$$d_i - d_j = w_{ij} \qquad \text{for each } (i, j) \in Q, \tag{18b}$$

$$w_{ij} \geq 0 \qquad \text{for each } (i, j) \in Q. \tag{18c}$$

This problem is an instance of the dual network flow problem.

Application 3: Time-Cost Tradeoff Problem in Project Scheduling

Project scheduling is regularly carried out in numerous industries. We will show that the *time-cost tradeoff problem*, an important problem in project scheduling, can be formulated as a dual network flow problem. We refer the reader to the book by Elmaghraby [1978] for a comprehensive discussion of the applications of network models in project scheduling.

We can envision a project as a directed graph G = (N, A) where the arc set A represents the jobs of the project and the node set N represents *events*, denoting the beginning and ending of jobs. Different jobs in the network have precedence relations. We require that all jobs directed into any node must be completed before any job directed out of the node begins. We let node s designate the beginning of the project and node T designate the ending of the project.

Normally, in project scheduling we assume that the job completion time is fixed; however, we here consider a more general settings where job completion times are variable. Let t_{ij} denote the time it takes to complete of job (i, j). We allow t_{ij} to vary in the range $[\alpha_{ij}, \beta_{ij}]$ and associate a convex cost function $F_{ij}(t_{ij})$ which captures the cost of completing the job for different completion times in the range $[\alpha_{ij}, \beta_{ij}]$. (These are often treated in the literature as "crash times.") We also have another set of decision variables υ_i's denoting *event times*; the variable υ_i denotes the time when jobs emanating from node i can begin. The time-cost tradeoff problem is to determine the minimum cost of completing the project in a specified time period T. This problem can be mathematically stated as follows:

$$\text{Minimize } \sum_{(i,j)\in A} F_{ij}(t_{ij}) \tag{19a}$$

subject to

$$\upsilon_t - \upsilon_s \leq T \tag{19b}$$

$$\upsilon_j - \upsilon_i \geq t_{ij} \qquad \text{for all } (i, j) \in A, \tag{19c}$$

$$\alpha_{ij} \leq t_{ij} \leq \beta_{ij} \qquad \text{for all } (i, j) \in A. \tag{19d}$$

Observe that the constraints (19b) capture the fact that the project must be completed within the time period T, and the constraints (19c) imply that for every arc (i, j) ∈ A, event j must occur at least w_{ij} time units later than the occurrence of event i. The formulation (19) is an instance of the dual network flow problem.

Application 4: Just-In-Time Scheduling in Project Management

Just-in-time is a management philosophy that has become quite popular in recent years. It attempts to eliminate waste by reducing slack times. We describe here an application of just-in-time scheduling applied to project management. This application has been adapted by us from Levner and Nemirovsky [1991]. As in the previous application, we denote a project by a directed graph G = (N, A), where set A denotes *jobs*, and the node set N denotes *events*. The network G also captures precedence relations among the arcs. We assume in this application that the completion times of all jobs are fixed. Let t_{ij} denote the time it takes to complete job (i, j). (Notice that t_{ij} is not a decision variable in this problem.) Let υ_i denote the time for event i.

Consider the feasible event times υ_i's that is, satisfying $\upsilon_j - \upsilon_i \geq t_{ij}$ for all (i, j) ∈ A. With respect to these event times, a job (i, j) will be completed at time $\upsilon_i + t_{ij}$ but the jobs emanating from node j will start at time υ_j. Let $w_{ij} = \upsilon_j - \upsilon_i - t_{ij}$ denote the *slack time,* and let $F_{ij}(w_{ij})$ denote its associated penalty cost. This penalty cost may capture the lost opportunity cost of the capital tied or some other factors (such as, perishability or deterioration in quality) which make slack times undesirable. There may also be some upper bounds β on slack times. We may assume without loss of generality that the lower bound on slack time is 0, since any other lower bound would be incorporated into the times to complete a task. The just-in-time project scheduling problem is to obtain event times υ_i's so that the project is completed within the specified time period T and the penalty cost associated with job slacks is minimum. This problem can be mathematically modeled as follows:

$$\text{Minimize } \sum_{(i,j)\in A} F_{ij}(w_{ij}) \tag{20a}$$

subject to

$$\upsilon_t - \upsilon_s \leq T \tag{20b}$$

$$\upsilon_j - \upsilon_i - w_{ij} = t_{ij} \qquad \text{for all } (i, j) \in A, \tag{20c}$$

$$0 \leq w_{ij} \leq \beta_{ij} \qquad \text{for all } (i, j) \in A. \tag{20d}$$

Application 5. Isotonic Regression Problem

The *isotonic regression problem* can be defined as follows. Given a set $A = \{a_1, a_2, \dots, a_n\} \in R^n$, find $X = \{x_1, x_2, \dots, x_n\} \in R^n$ so as to

$$\text{Minimize } \sum_{j=1}^{n} B_j(x_j - a_j) \tag{21a}$$

subject to

$$x_j \leq x_{j+1} \qquad \text{for all } j = 1, 2, \dots, n\text{-}1, \tag{21b}$$

$$l_j \leq x_j \leq u_j \qquad \text{for all } j = 1, 2, \dots, n\text{-}1, \tag{21c}$$

$$x_j \text{ integer} \qquad \text{for all } j = 1, 2, \dots, n\text{-}1. \tag{21d}$$

where $B_j(x_j - a_j)$ is a convex function for every j, $1 \leq j \leq n$. The isotonic regression problem arises in statistics, production planning, and inventory control (see for example, Barlow et al. [1972] and Robertson et al. [1988]). As an illustration of the isotonic regression, consider a full tank where fuel is being consumed at a slow rate and measurements of the fuel take are being taken at different points in time. Suppose these measurements are a_1, a_2, .., a_n. Due to the errors in the measurements, these numbers may not be in the nonincreasing order despite the fact that the true amounts of fuel remaining in the tank are nonincreasing. However, we need to determine these measurements as accurately as possible. One possible way to accomplish this could be to perturb these numbers to $x_1 \geq x_2 \geq \dots \geq x_n$ so that the cost of perturbation given by $\sum_{j=1}^{n} B_j(x_j - a_j)$ is minimum, where each $B_j(x_j - a_j)$ is a convex function. We can transform this problem to the isotonic regression problem by replacing x_j's by their negatives.

If we define $P = \{1, 2, \dots, n\}$ and $Q = \{(j, j+1) : j = 1, 2, \dots, n\text{-}1\}$ and require that x_j must be integer, then the isotonic regression problem can be cast as a dual network flow problem. However, it is a very special case of the dual network flow problem and more efficient algorithms can be developed to solve it compared to the dual network flow problem. Ahuja and Orlin [1998] recently developed an O(n log U) algorithm to solve the isotonic regression problem. Hochbaum and Queyranne [1999] described an algorithm for the isotonic regression of complexity O(n + n log U) where the second term in the complexity expression is the work required to find the integer minima of functions in the objective. These functions are in general n convex functions in which case the complexity is as stated. In some cases the functions are simpler and the second term can be implemented more efficiently.

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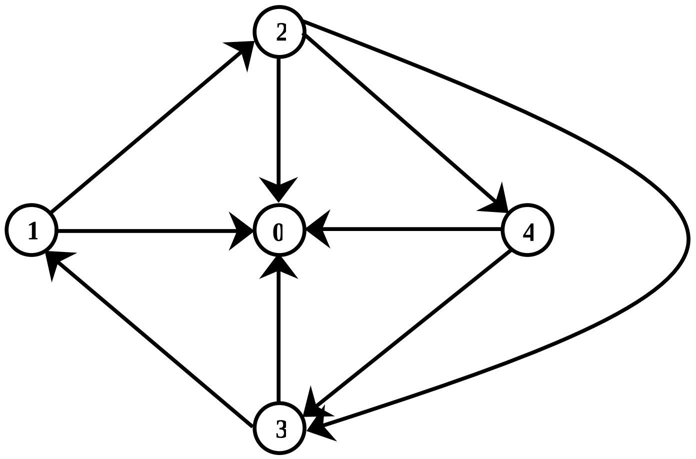

Figure 1. Illustrating the transformation.

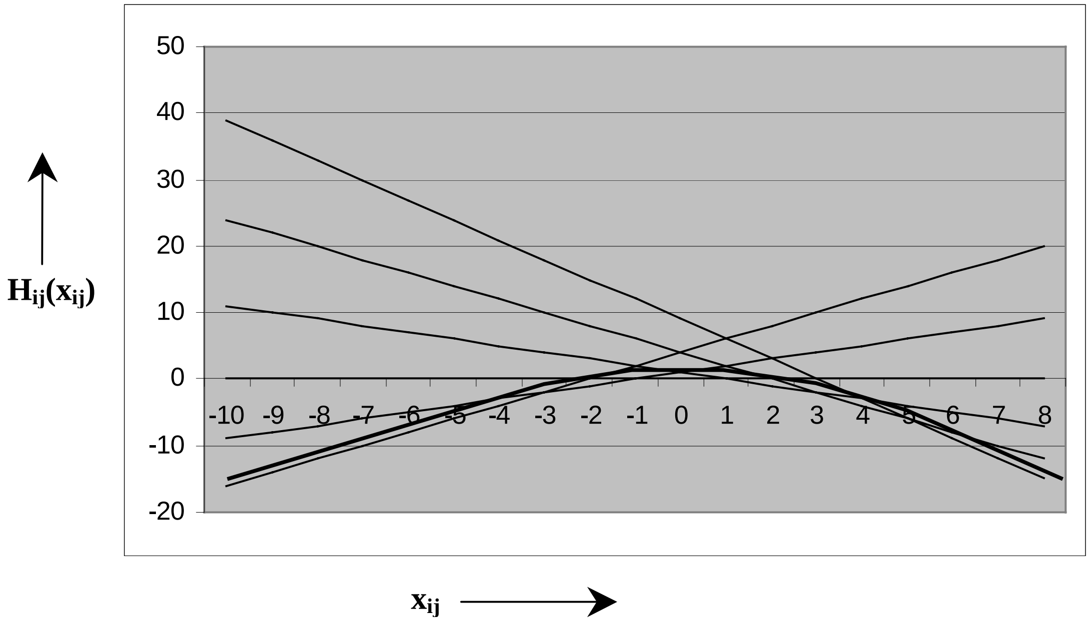

Figure 2. An example of the function