Noise in Switched-Capacitor Circuits

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What you will learn...

• How to analyze noise in switched-capacitor circuits

• Significance of switch noise vs. OTA noise
  Power efficient solution
  Impact of OTA architecture

• Design example for $\Delta \Sigma$ modulator
Review

• Previous analysis of kT/C noise (ignoring OTA/opamp noise)
  Phase 1: kT/C₁ noise (on each side)
  Phase 2: kT/C₁ added to previous noise (on each side)
  Total Noise (input referred): 2kT/C₁
  Differentially: 4kT/C₁

Review

• SNR (differential)
  Total noise power: 4kT/C₁
  Signal power: (2V)²/2
  SNR: V²C₁/2kT

• SNR (single-ended)
  Total noise power: 2kT/C₁ (sampling capacitor C₁)
  Signal power: V²/2 (signal from -V to V)
  SNR: V²C₁/4kT
Noise in an Integrator

- Two noise sources $V_{C1}$ and $V_{OUT}$
  - $V_{C1}$: Represents input-referred sampled noise on input switching transistors + OTA
  - $V_{OUT}$: Represents output-referred (non-sampled) noise from OTA

![Noise in an Integrator Diagram]

Thermal Noise in OTAs

- Single-Ended Example
  - Noise current from each transistor is $\bar{i}_n^2 = 4kT\gamma g_m$
  - Assume $\gamma = 2/3$

![Thermal Noise in OTAs Diagram]
Thermal Noise in OTAs

- Single-Ended Example
  Thermal noise in single-ended OTA
  Assuming paths match, tail current source \( M_5 \) does not contribute noise to output
  PSD of noise voltage in \( M_1 \) (and \( M_2 \)): \( \frac{8}{3} \frac{kT}{g_{m1}} \)

  PSD of noise voltage in \( M_3 \) (and \( M_4 \)): \( \frac{8}{3} \frac{kTg_{m3}}{g_{m1}^2} \)

  Total input referred noise from \( M_1 \) - \( M_4 \)
  \[
  S_{n,eq} = \frac{16}{3} \frac{kT}{g_{m1}} \left( 1 + \frac{g_{m3}}{g_{m1}} \right) = \frac{16}{3} \frac{kT}{g_{m1}} n_f
  \]

  Noise factor \( n_f \) depends on architecture

OTA with capacitive feedback

- Analyze output noise in single-stage OTA
  Use capacitive feedback in the amplification / integration phase of a switched-capacitor circuit
OTA with capacitive feedback

• Transfer function of closed loop OTA

$$H(s) = \frac{V_{\text{OUT}}}{V_{n,\text{eq}}} = \frac{G}{1 + \frac{s}{\omega_o}}$$

where the DC Gain and 1st-pole frequency are

$$G \approx \frac{1}{\beta} = 1 + \frac{C_1}{C_2} \quad \omega_o = \frac{\beta g_{m1}}{C_o}$$

Load capacitance $C_O$ depends on the type of OTA – for a single-stage, it is $C_1 + C_1C_2/(C_1 + C_2)$, while for a two-stage, it is the compensation capacitor $C_C$.

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OTA with capacitive feedback

• Integrate total noise at output

$$\overline{V^2_{\text{OUT}}} = \int_0^\infty S_{n,\text{eq}}(f)|H(j2\pi f)|^2 df$$

$$= \frac{16}{3} \frac{kT}{g_{m1}} n_f \frac{\omega_o}{4} G^2$$

$$= \frac{4kT}{3\beta C_o} n_f$$

Minimum output noise for $\beta=1$ is $\frac{4kT}{3C_o} n_f$

Not a function of $g_{m1}$ since bandwidth is proportional to $g_{m1}$ while PSD is inversely proportional to $g_{m1}$.
OTA with capacitive feedback

• Graphically...

\[ S_{\text{eq}}(f) = \frac{16kTn}{3g_m} \]

\[ H(s) \]

\[ H_{\text{brick}}(s) \]

1/\beta

\[ f - \frac{\pi f_o}{2} \]

Noise is effectively filtered by equivalent brick wall response with cut-off frequency \( \pi f_o/2 \) (or \( \omega_o/4 \) or \( 1/4\pi \))

Total noise at \( V_{\text{OUT}} \) is the integral of the noise within the brick wall filter (area is simply \( \pi f_o/2 \times 1/\beta^2 \))

Sampled Thermal Noise

• What happens to noise once it gets sampled?

Total noise power is the same

Noise is aliased – folded back from higher frequencies to lower frequencies

PSD of the noise increases significantly
Sampled Thermal Noise

- Same total area, but PSD is larger from 0 to $f_s/2$

$$S_{vout}(f) = \frac{G^2 S_{n,eq}}{4\pi f_s/2} = \frac{V_{OUT}^2}{f_s/2} = \frac{4kT}{3\beta C_o n_f} \frac{1}{f_s/2}$$

Low frequency PSD $G^2 S_{n,eq}$ is increased by $\frac{1}{2\pi f_s} = \frac{\pi f_{3dB}}{f_s}$

Sampled Thermal Noise

- $1/f_{3dB}$ is the settling time of the system, while $1/2f_s$ is the settling period for a two-phase clock

$$e^{1/2f_s} \tau < 2^{-(N+1)}$$

$$\frac{\pi f_{3dB}}{f_s} > (N+1)\ln2$$

PSD is increased by at least $(N+1)\ln2$

If $N = 10$ bits, PSD is increased by 7.6, or 8.8dB

- This is an inherent disadvantage of sampled-data compared to continuous-time systems

  But noise is reduced by oversampling ratio after digital filtering
Noise in a SC Integrator

- Using the parasitic-insensitive SC integrator

- Two phases to consider
  1) Sampling Phase
     - Includes noise from both $\phi_1$ switches
  2) Integrating Phase
     - Includes noise from both $\phi_2$ switches and OTA

Noise in a SC Integrator

- Phase 1: Sampling

Noise PSD from two switches: $S_{Ron}(f) = 8kTR_{ON}$
Time constant of R-C filter: $\tau = 2R_{ON}C_1$
PSD of noise voltage across $C_1$

$$S_{C_1}(f) = \frac{8kTR_{ON}}{1 + (2\pi f\tau)^2}$$
Noise in a SC Integrator

• Phase 1: Sampling

Integrated across entire spectrum, total noise power in $C_1$ is

$$V_{C1,sw1}^2 = \frac{8kTR_{ON}}{4\tau} = \frac{kT}{C_1}$$

Independent of $R_{ON}$ (PSD is proportional to $R_{ON}$, bandwidth is inversely proportional to $R_{ON}$)

After sampling, charge is trapped in $C_1$

Noise in a SC Integrator

• Phase 2: Integrating

• Two noise sources: switches and OTA

Noise PSD from two switches: $S_{Ron}(f) = 8kTR_{ON}$

Noise PSD from OTA: $S_{vn,eq}(f) = \frac{16}{3} \frac{kT}{g_{m1}} n_f$

Noise power across $C_1$ charges to $2V_{Ron}^2 + V_{n,eq}^2$
Noise in a SC Integrator

• What is the time-constant?

\[
\tau = \frac{1}{sC_2 + R_L} \frac{1}{1 + g_{m_1}R_L}
\]

For large \( R_L \), assume that

\[
Z_{in} \approx \frac{1}{g_{m_1}}
\]

Resulting time constant

\[
\tau = \left(2R_{ON} + \frac{1}{g_{m_1}}\right)C_1
\]

Noise in a SC Integrator

• Total noise power with both switches and OTA on integrating phase

\[
\overline{V^2_{C_{1,op}}} = \frac{S_{m,eq}(f)}{4\tau} = \frac{16kT}{3g_{m_1}} \frac{n_f}{4(2R_{ON} + 1/g_{m_1})C_1} = \frac{4kT}{3C_1} \frac{n_f}{(1+x)}
\]

\[
\overline{V^2_{C_{1,sw2}}} = \frac{S_{Ron}(f)}{4\tau} = \frac{8kTR_{ON}}{4(2R_{ON} + 1/g_{m_1})C_1} = \frac{kT}{C_1} \frac{x}{(1+x)}
\]

Introduced extra parameter

\[
x = 2R_{ON}g_{m_1}
\]
Noise in a SC Integrator

• Total noise power on C1 from both phases

\[
\bar{V}_{C1}^2 = \bar{V}_{C1,op}^2 + \bar{V}_{C1,sw1}^2 + \bar{V}_{C1,sw2}^2
\]

\[
= \frac{4kT}{3C_1} \frac{n_f}{1+x} + \frac{kT}{C_1} \frac{x}{1+x} + \frac{kT}{C_1}
\]

\[
= \frac{kT}{C_1} \left( \frac{4n_f}{3} + 1 + 2x \right)
\]

Lowest possible noise achieved if \( x \to \infty \)

In this case, \( \bar{V}_{C1}^2 = \frac{2kT}{C_1} \)

What was assumed to be the total noise was actually the least possible noise!

Noise Contributions

• Percentage noise contribution from switches and OTA (assume \( n_f = 1.5 \))

\[ x = 2R_{ON}g_{m1} \]
Noise Contributions

- When $g_{m1} >> 1/R_{ON}$ ($x >> 1)$...
  Switch dominates both bandwidth and noise
  Total noise power is minimized

- When $g_{m1} << 1/R_{ON}$ ($x << 1)$...
  OTA dominates both bandwidth and noise
  Power-efficient solution
  Minimize $g_{m1}$ (and power) for a given settling
  time and noise

\[
g_{m1} = \frac{kT}{rV_{C1}^2} \left( \frac{4}{3} n_f + 1 + 2x \right)
\]

Minimized for $x=0$

Maximum Noise

- How much larger can the noise get?
  Depends on $n_f$... (table excludes cascode noise)

<table>
<thead>
<tr>
<th>Architecture</th>
<th>Relative $V_{EFF}$s</th>
<th>$n_f$</th>
<th>Maximum Noise ($x=0$)</th>
<th>$+dB$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Telescopic/Diff.Pair</td>
<td>$V_{EFF,1}=V_{EFF,n}/2$</td>
<td>1.5</td>
<td>$3\cdot kT/C_1$</td>
<td>1.76</td>
</tr>
<tr>
<td>Telescopic/Diff.Pair</td>
<td>$V_{EFF,1}=V_{EFF,n}$</td>
<td>2</td>
<td>$3.67\cdot kT/C_1$</td>
<td>2.63</td>
</tr>
<tr>
<td>Folded Cascode</td>
<td>$V_{EFF,1}=V_{EFF,n}/2$</td>
<td>2.5</td>
<td>$4.33\cdot kT/C_1$</td>
<td>3.36</td>
</tr>
<tr>
<td>Folded Cascode</td>
<td>$V_{EFF,1}=V_{EFF,n}$</td>
<td>4</td>
<td>$6.33\cdot kT/C_1$</td>
<td>5.01</td>
</tr>
</tbody>
</table>
Separate Input Capacitors

• Using separate input caps increases noise
  Each additional input capacitor adds to the total noise
  Separate caps help reduce signal dependent disturbances in the DAC reference voltages

\[ V_{c1}^2 = \frac{kT}{C_1} \left( \frac{4n_r / 3 + 1 + 2x}{1 + x} \right) \left( 1 + \frac{C_{ia}}{C_1} + \ldots \right) \]

Differential vs. Single-Ended

• All previous calculations assumed single-ended operation
  For same settling time, \( g_{m1,2} \) is the same, resulting in the same total power [0dB]
  Differential input signal is twice as large [gain 6dB]
  Differential operation has twice as many caps and therefore twice as much capacitor noise (assume same size per side – \( C_1 \) and \( C_2 \)) [lose ~1.2dB for \( n_f=1.5, x=0 \ldots \) less for larger \( n_f \)]

• Net Improvement: ~4.8dB
Differential vs. Single-Ended

- **Single-Ended Noise**
  \[
  \bar{V}_{C1,se}^2 = \frac{kT}{C_1} \left( \frac{4n_f / 3 + 1 + 2x}{1 + x} \right)
  \]

- **Differential Noise**
  \[
  \bar{V}_{C1,diff}^2 = \bar{V}_{C1,op}^2 + \bar{V}_{C1,sw1}^2 + \bar{V}_{C1,sw2}^2 = 4kT \frac{n_f}{3C_1 (1 + x)} + 2kT \frac{x}{C_1 (1 + x)} + \frac{2kT}{C_1}
  \]
  \[
  \bar{V}_{C1,diff}^2 = \frac{kT}{C_1} \left( \frac{4n_f / 3 + 2 + 4x}{1 + x} \right)
  \]

- **Relative Noise** (for \(n_f=1.5, x=0\))
  \[
  \frac{\bar{V}_{C1,diff}^2}{\bar{V}_{C1,se}^2} = \frac{4n_f / 3 + 2 + 4x}{4n_f / 3 + 1 + 2x} = \frac{4}{3}
  \]

Noise in an Integrator

- **What is the total output-referred noise in an integrator?**

Assume an integrator transfer function

\[
H(z) = \frac{kz^{-1}}{1 + \mu(1 + k) - (1 + \mu)z^{-1}} \approx \frac{kz^{-1}}{1 - z^{-1}}
\]

where \(k = \frac{C_1}{C_2}\) and \(\mu = \frac{1}{A}\)
Noise in an Integrator

• Total output-referred noise PSD

\[ S_{\text{INT}}(f) = S_{C1}(f)|H(z)|^2 + S_{\text{OUT}}(f) \]

where

\[ \overline{V^2_{\text{OUT}}} = \frac{4kT}{3\beta C_o} n_t \]

and

\[ \overline{V^2_{C1}} = \frac{kT}{C_1} \left( \frac{4n_t / 3 + 1 + 2x}{1 + x} \right) \]

Since all noise sources are sampled, white PSDs

\[ S_x = \frac{V^2_x}{f_s / 2} \]

To find output-referred noise for a given OSR in a ΔΣ modulator:

\[ \overline{V^2_{\text{INT}}} = \int_0^{f_s/(2\cdot\text{OSR})} S_{\text{INT}}(f) df \]

Noise in a ΔΣ Modulator

• How do we find the total input-referred noise in a ΔΣ modulator?

1) Find all thermal noise sources
2) Find PSDs of the thermal noise sources
3) Find transfer functions from each noise source to the output
4) Using the transfer functions, integrate all PSDs from DC to the signal band edge \( f_s/2 \cdot \text{OSR} \)
5) Sum the noise powers to determine the total output thermal noise
6) Input noise = output noise (assuming STF is ~1 in the signal band)
Noise in a $\Delta\Sigma$ Modulator

- Example
  
  $f_s = 100\text{MHz}$, $T = 10\text{ns}$, OSR = 32
  
  SNR = 80dB (13-bit resolution)
  
  Input Signal Power = $0.25V^2$ (-6dB from $1V^2$)
  
  Noise Budget: 75% thermal noise
  
  Total input referred thermal noise:

1) Find all thermal noise sources

\[
\begin{align*}
\overline{V_{n1}^2} &= \frac{kT}{C_{1A}} \left( \frac{4n_{fA} / 3 + 1 + 2x_A}{1 + x_A} \right) \\
\overline{V_{n2}^2} &= \frac{kT}{C_{1B}} \left( \frac{4n_{fB} / 3 + 1 + 2x_B}{1 + x_B} \right) \\
\overline{V_{no1}^2} &= \frac{4kT}{3\beta_A C_{OA}} n_{fA} \\
\overline{V_{no2}^2} &= \frac{4kT}{3\beta_B C_{OB}} n_{fB} \\
\overline{V_{n3}^2} &= \frac{2kT}{C_{f1}} \left( 1 + \frac{C_{f2}}{C_{f1}} + \frac{C_{f3}}{C_{f1}} \right) = \frac{2kT}{C_{f1}} (1 + 2 + 1)
\end{align*}
\]
Noise in a $\Delta\Sigma$ Modulator

2) Find PSDs of the thermal noise sources
   For each of the mean square voltage sources,
   \[
   S_x = \frac{V_x^2}{f_s/2}
   \]

3) Find transfer functions from each noise source to the output
   Assume ideal integrators
   \[
   H_A(z) = H_B(z) = \frac{z^{-1}}{1 - z^{-1}}
   \]
   \[
   STF(z) = 1
   \]
   \[
   NTF(z) = (1 - z^{-1})^2 = \frac{1}{1 + 2H(z) + H(z)^2}
   \]

Noise in a $\Delta\Sigma$ Modulator

3) Find transfer functions from each noise source to the output
   From input of $H_A(z)$ to output...
   \[
   NTF_{i1}(z) = \left(2H(z) + H(z)^2\right)NTF(z)
   \]
   \[
   = \frac{2H(z) + H(z)^2}{1 + 2H(z) + H(z)^2} = 2z^{-1} - z^{-2}
   \]

   From output of $H_A(z)$ to output...
   \[
   NTF_{o1}(z) = (2 + H(z))NTF(z)
   \]
   \[
   = \frac{2 + H(z)}{1 + 2H(z) + H(z)^2} = (1 - z^{-1})(2 - z^{-1})
   \]
3) Find transfer functions from each noise source to the output

From input of $H_B(z)$ to output…

$$NTF_{i2}(z) = H(z)NTF(z)$$

$$= \frac{H(z)}{1+2H(z)+H(z)^2} = z^{-1}(1 - z^{-1})$$

From output of $H_B(z)$ to output (equal to transfer function at input of summer to output)…

$$NTF_{o2}(z) = NTF(z) = (1 - z^{-1})^2$$
4) Using the transfer functions, integrate all PSDs from DC to the signal band edge $f_s/2 \cdot \text{OSR}$

Use MATLAB/Maple to solve the integrals...

$$\overline{N_{i1}^2} = \frac{V_{n1}^2}{f_s/2} \int_0^{f_s/(2 \cdot \text{OSR})} |NTF_{i1}(f)|^2 \, df$$

$$= \frac{V_{n1}^2}{f_s/2} \left[ \frac{5f_s}{2 \cdot \text{OSR}} - \frac{2f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \right]$$

$$\overline{N_{o1}^2} = \frac{V_{no1}^2}{f_s/2} \int_0^{f_s/(2 \cdot \text{OSR})} |NTF_{o1}(f)|^2 \, df$$

$$= \frac{V_{no1}^2}{f_s/2} \left[ \frac{7f_s}{\text{OSR}} + \frac{2f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \cos \left( \frac{\pi}{\text{OSR}} \right) - \frac{9f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \right]$$

4) Using the transfer functions, integrate all PSDs from DC to the signal band edge $f_s/2 \cdot \text{OSR}$

$$\overline{N_{i2}^2} = \frac{V_{n2}^2}{f_s/2} \left[ \frac{f_s}{\text{OSR}} - \frac{f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \right]$$

$$\overline{N_{o2}^2} = \frac{V_{no2}^2 + V_{n3}^2}{f_s/2} \left[ \frac{3f_s}{\text{OSR}} + \frac{f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \cos \left( \frac{\pi}{\text{OSR}} \right) - \frac{4f_s}{\pi} \sin \left( \frac{\pi}{\text{OSR}} \right) \right]$$

(Some simplifications can be made for large OSR)
5) Sum the noise powers to determine the total output thermal noise

Assume $x_A = x_B = 0.1$ and $n_{fA} = n_{fB} = 1.5$

\[
\overline{V_{TH}^2} \approx \frac{2.9kT}{C_{1A}} \frac{1}{OSR} + \frac{2kT}{\beta_A C_{OA}} \frac{\pi^2}{3OSR^3} + \frac{2.9kT}{C_{1B}} \frac{\pi^2}{3OSR^3} + \frac{2kT}{\beta_B C_{OB}} \frac{\pi^4}{5OSR^5} + \frac{8kT}{C_{f1}} \frac{\pi^4}{5OSR^5}
\]

With an OSR of 32, first term is most significant (assume $\beta_A = \beta_B = 1/3$)

\[
\overline{V_{TH}^2} \approx 9.1 \times 10^{-2} \frac{kT}{C_{1A}} + 6.0 \times 10^{-4} \frac{kT}{C_{OA}} + 2.9 \times 10^{-4} \frac{kT}{C_{1B}} + \ldots
\]

6) Input noise = output noise (assuming STF is ~1 in the signal band)

\[
\overline{V_{TH}^2} \approx 9.1 \times 10^{-2} \frac{kT}{C_{1A}} = (43.4 \mu V)^2
\]

=> $C_{1A} = 200 \text{fF}$

Assuming other capacitors are smaller than $C_{1A}$, then subsequent terms are insignificant and the approximation is valid.

If lower oversampling ratios are used, other terms may become more significant in the calculation.
Noise in a Pipeline ADC

- Similar procedure to $\Delta \Sigma$ modulator, except transfer functions are much easier to compute.

- Differences...
  - Input refer all noise sources
  - Gain from each stage to the input is a scalar
  - Noise from later stages will be more significant since typical stage gains are as low as 2
  - Sample-and-Hold adds extra noise which is input referred with a gain of 1
  - Entire noise power is added since the signal band is from 0 to $f_s/2$ (OSR=1)

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Noise in a Pipeline ADC

- Example
  - If each stage has a gain $G_1$, $G_2$, ..., $G_N$

\[
\overline{N_i^2} = \frac{V_{ni1}^2}{G_1^2} + \frac{V_{no1}^2 + V_{ni2}^2}{G_1^2 G_2^2} + \frac{V_{no2}^2 + V_{ni3}^2}{G_1^2 G_2^2 G_3^2} + \cdots + \frac{V_{noN}^2}{G_1^2 G_2^2 \cdots G_N^2}
\]

S/H stage noise will add directly to $V_{ni1}$

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![Diagram of Pipeline ADC](image)
Further Reading

- Appendix C of *Understanding Delta-Sigma Data Converters*, Schreier and Temes
- Schreier et al., *Design-Oriented Estimation of Thermal Noise in Switched-Capacitor Circuits*, TCAS-I, Nov. 2005