Frequency Response

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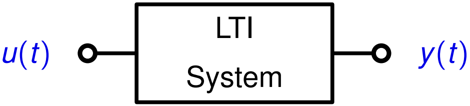
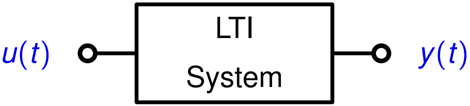
LTI System



- An Linear Time-Invariant (LTI) system must be both... LINEAR and TIME-INVARIANT
- LTI systems are important as they allow us to define...
  - Frequency response
  - impulse/step response
  - a relation between the impulse response and freq response
- From now on, we will assume all systems are LTI systems unless otherwise stated

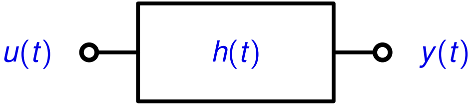
LTI System



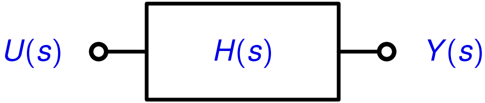
- A system is LINEAR if and only if ...
  - If $u_1(t)$ results in $y_1(t)$
    and $u_2(t)$ results in $y_2(t)$
  - then $u_1(t) + u_2(t)$ results in $y_1(t) + y_2(t)$
    for all $u_1(t)$,$u_2(t)$
  - In other words, if you sum 2 different inputs, the output should be the sum of the 2 different outputs
- A system is TIME-INVARIANT if and only if ...
  - If $u_1(t)$ results in $y_1(t)$
  - then $u_1(t-\tau)$ results in $y_1(t-\tau)$
    for all $u_1(t), \tau$
  - In other words, if you shift the input in time by any amount, the output should also shift in time by the same amount

Impulse/Freq Response

- We characterize an LTI system by
  - its impulse response, $h(t)$
  - its frequency response $H(s)$
  - where $H(s)$ is the Laplace transform of $h(t)$
    $U(s)/Y(s)$ is the Laplace transform of $u(t)/y(t)$
- In the time domain



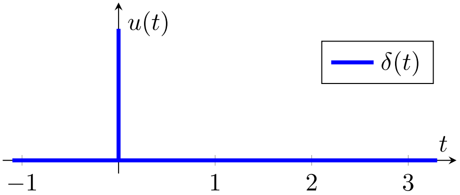
- In the freq domain



$H(s) \equiv \mathcal{L}[h(t)]$ where $\mathcal{L}$ is the Laplace transform

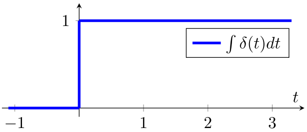
Impulse Response

- Impulse response of a system
  - the output, $y(t)$, when $u(t) = \delta(t)$
    where $\delta(t)$ is the Dirac Delta function



  - area of impulse is 1

- If you take the integral of the impulse function



- $h(t) = y(t)$ when $u(t) = \delta(t)$

Impulse Response

- Given the impulse response, $h(t)$ for a system
- Can find $y(t)$ for an any $u(t)$ as

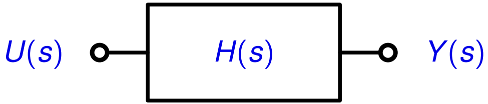
  $y(t) = h(t) \circledast u(t)$
  where $\circledast$ is the convolution operator
- Can find $Y(s)$ for any $U(s)$ as

  $Y(s) = H(s)U(s)$
- Convolution in the time domain is multiplication in the freq domain
- For sinusoidal inputs, we make use of the transfer-function, $H(s)$ where $H(s) \equiv \mathcal{L}[h(t)]$

Transfer-function, $H(s)$

- $H(s)$ is a ratio of 2 polynomials in "s" IF
  - system is made up of lumped elements such as resistors, capacitors, inductors, transistors, etc
  - Example

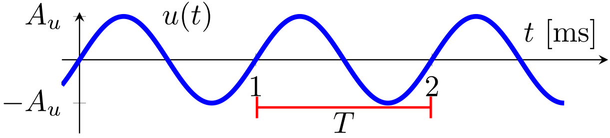
    $H(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_3 s^3 + b_2 s^2 + b_1 s + b_0}$
- $H(s)$ is NOT a ratio of 2 polynomials in "s" for many other systems such as ...
  - transmission line
  - distributed RC wire



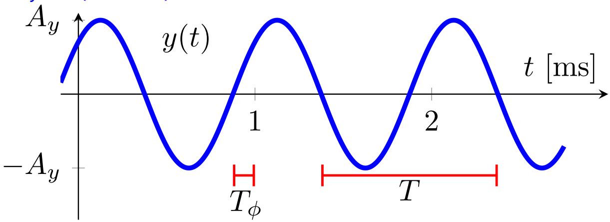
- $H(s)$ is the frequency response for a system
- $H(s)$ tells us how the system will affect sinusoidal input signals
  - A sinusoidal input signal of frequency $\omega\ \mathrm{rad/s}$ will result in an output sinusoidal signal at the same frequency
  - $\omega = 2\pi f$ where f is the freq in Hertz
  - However, the amplitude and phase of the output signal may be changed relative to the input signal.
  - $H(s)$ tells us this magnitude/phase change
  - We let $s = j\omega$ and find $|H(j\omega)|$ and $\angle H(j\omega)$
  - $H(j\omega)$ is a complex number

Transfer-function, $H(s)$

- Input sinusoid
  - $u(t) = A_u \sin(\omega t) = A_u \sin(2\pi f t)$



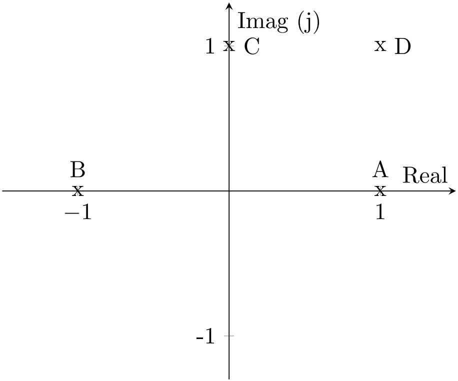
- Output sinusoid
  - $y(t) = A_y \sin(\omega t + \phi)$



Transfer-function, $H(s)$

- $T = 1/f = 2\pi/\omega$ (sinusoid period)
- $T_\phi = (\phi/2\pi)T$ (sinusoid phase shift)
- Above example
  - $T = 1\ \mathrm{ms}$
  - $f = 1\ \mathrm{kHz} \qquad \omega = 2\pi \times 1000\ \mathrm{rad/s}$
  - $T_\phi = 1/8\ \mathrm{ms} \qquad \phi = \pi/4\ \mathrm{rad}$
- Magnitude Response
  - $A_y/A_u = |H(j\omega)|$
- Phase Response
  - $\phi = \angle H(j\omega)$

Complex Numbers



$A = 1$ $\quad |A| = 1$ $\quad \angle A = 0^\circ$

$B = -1$ $\quad |B| = 1$ $\quad \angle B = 180^\circ$

$C = j$ $\quad |C| = 1$ $\quad \angle C = 90^\circ$

$D = 1 + j$ $\quad |D| = \sqrt{2}$ $\quad \angle D = 45^\circ$

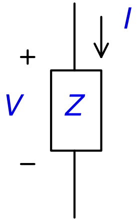
Complex Numbers

- $z = a + jb$
  - z complex
  - a, b are real
- $|z| = \sqrt{(a^2 + b^2)}$
- $\angle z = \tan^{-1}(b/a) \quad$ if $a > 0$
- $\angle z = \tan^{-1}(b/a) + \pi \quad$ if $a < 0$
- Can write z in polar form (better for multiplication/division)
- $z = |z|e^{j\angle z}$
- $z = \frac{z_1}{z_2}$

  $|z| = |z_1|/|z_2|$

  $\angle z = \angle z_1 - \angle z_2$

Ohm's Law with Impedances



$I = \frac{V}{Z}$

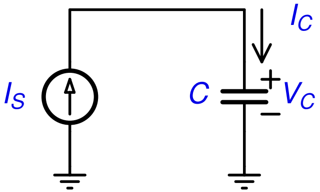
Resistor of size R $\quad Z = R$

Capacitor of size C $\quad Z = 1/sC$

Inductor of size L $\quad Z = sL$

- "s" is the Laplace transform variable
- We let $s = j\omega$ to evaluate what happens for a sinusoidal signal at frequency $\omega \ \mathrm{rad/s}$

Capacitor Impedance



$$I_C = I_S$$

- Given $I_S = A_I \sin(\omega t)$
- We want to find $V_C = A_C \sin(\omega t + \phi)$ across the capacitor
- Want to find A_C and ϕ
- Define $H(s) \equiv \frac{V_C(s)}{I_C(s)}$

Capacitor Impedance

- From Ohm's Law, we have
  $I_C(s) = V_C(s)/Z_C$
  resulting in
  $H(s) = V_C(s)/I_C(s) = Z_C$
  Since $Z_C = 1/sC$, we have
  $H(s) = 1/sC$
- Magnitude response
  - $|H(j\omega) = \left|\frac{1}{j\omega C}\right| = \frac{1}{\omega C}$
  - $|V_C(s)| = \left(\frac{1}{\omega C}\right) \times |I_C(s)|$
  - $A_V = \frac{A_I}{\omega C}$
- Phase response
  - $\angle H(j\omega) = \angle\left(\frac{1}{j\omega C}\right) = \angle(1) - \angle(j\omega C)$
    $\angle H(j\omega) = 0^\circ - 90^\circ = -90^\circ$
  - So the capacitor voltage sinusoid is always shifted by 90° compared to the capacitor current sinusoidal signal for all frequencies

Capacitor Impedance

- Example
  - $A_I = 1\ \mathrm{mA}$
    $f = 1\ \mathrm{kHz} \;\Rightarrow\; \omega = 6.283\ \mathrm{krad/s}$
    $C = 1\ \mu\mathrm{F}$
- $A_V = A_I/(\omega C) = (1e-3)/(6.283e3 \times 1e-6)$
  $A_V = 0.159\ \mathrm{V}$
- $\phi = -90^\circ$
- So the peak cap voltage is $0.159\ \mathrm{V}$ and the cap voltage sinusoid leads the cap current by 90°

Transfer-Function of LTI System

- We will restrict ourselves to
  - Real-valued impulse response (some wireless systems make use of complex input/output signals)
  - Circuit with lumped elements (resistors, capacitors, inductors, independent/dependent voltage/current sources)
- Polynomial Form for $H(s)$

$$H(s) = \frac{a_o + a_1 s + a_2 s^2 + ... + a_m s^m}{1 + b_1 s + b_2 s^2 + ... + b_n s^n} \tag{1}$$

- Root Form for $H(s)$

$$H(s) = \left(\frac{a_m}{b_n}\right) \frac{(s + z_1)(s + z_2)...(s + z_m)}{(s + \omega_1)(s + \omega_2)...(s + \omega_n)} \tag{2}$$

  - Shows the zeros ($-z_i$) and poles ($-\omega_i$)

Transfer-Function of LTI System

- Zeros
  - Values of s where $H(s) = 0$
  - Zero at $s = -z_1 \;\Rightarrow H(-z_1) = 0$
- Poles
  - Values of s where $H(s) \to \infty$
  - pole at $s = -\omega_1 \;\Rightarrow H(-\omega_1) \to \infty$
- In general poles and zeros can be complex values
  - for real-valued transfer-functions, complex poles (and zeros) will occur in complex conjugate pairs.
  - Real-valued transfer-functions... all a_i, b_i are real
  - Complex conjugate pairs... if $x + jy$ is a pole (or zero) then $x - jy$ is also a pole (or zero) for $y \neq 0$

Stability of LTI System

- 3 types of stability for a system
- Strictly STABLE
  - Integral of magnitude of impulse response is finite
- Marginally STABLE
  - Integral of magnitude of impulse response grows at most linear in time
- UNSTABLE
  - Integral of magnitude of impulse response grows faster than linear in time

- Can also look at transfer-function to determine stability of a system

Stability of LTI System

- Strictly STABLE
  - $n \geq m$
  - All poles are in the left half plane
- Marginally STABLE
  - $n \geq m$
  - Has at least one pole on $j\omega$ axis and other poles in the left half plane
- UNSTABLE (if either condition occurs)
  - At least one pole in the right half plane

    OR
  - If $n < m$

    (high freq gain grows without bound)

Real Valued Poles/Zeros

- For this freq analysis, all poles/zeros will be real valued unless otherwise stated.
  - This is the case for LTI circuits **without feedback or inductors**.
  - In other words, all poles/zeros will occur on the real axis.
  - So all z_i and ω_i are real valued
- Examples

$H(s) = \frac{1}{s+2}$
zeros: none
poles: $\omega_1 = -2$

$H(s) = \frac{s}{s+3}$
zeros: $z_1 = 0$
poles: $\omega_1 = -3$

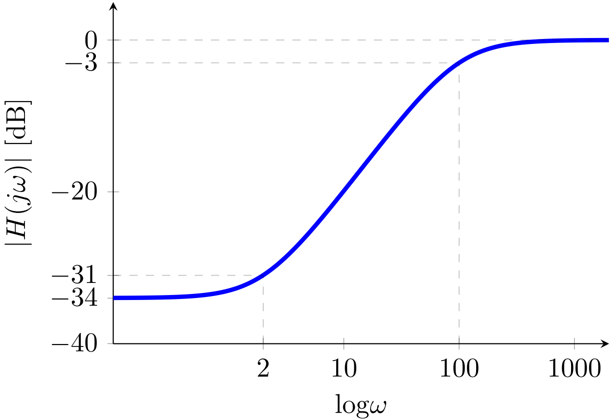
$H(s) = \frac{s(s-2)}{(s+1)(s+3)}$
zeros: $z_1 = 0 \quad z_2 = 2$
poles: $\omega_1 = -1 \quad \omega_2 = -3$

Real Valued Poles/Zeros

$$H(s) = \left(\frac{a_m}{b_n}\right) \frac{(s+z_1)(s+z_2)...(s+z_m)}{(s+\omega_1)(s+\omega_2)...(s+\omega_n)}$$

- For above
  - poles at $-\omega_i$
- HOWEVER, often said pole frequency is at ω_i
  - Since if we only consider pole
  - $|H(s)|$ is decreased by $\sqrt{2}$ at $H(j\omega_i)$
- Example
  - $H(s) = \frac{1}{s+1}$
  - $|H(0)| = 1 \quad |H(j1)| = \frac{1}{\sqrt{2}} \quad |H(j\infty)| = 0$
  - So for $\omega = 1\ \mathrm{rad/s}$, $|H(s)|$ decreased by $\sqrt{2}$
- Similar for zeros, often said zero frequency is at z_i
  - $|H(s)|$ increased by $\sqrt{2}$ at zero freq

- Example
  - $H(s) = \frac{s+2}{s+100}$
  - $|H(0)| = 0.02 \qquad |H(j\infty)| = 1$
  - $|H(j2)| \approx \sqrt{2}|H(0)|$ a 3dB increase
  - $|H(j100)| \approx (1/\sqrt{2})|H(j\infty)|$ a 3db decrease



Real Valued Poles/Zeros

$$H(s) = \left(\frac{a_m}{b_n}\right)\frac{(s+z_1)(s+z_2)...(s+z_m)}{(s+\omega_1)(s+\omega_2)...(s+\omega_n)}$$

- From above, can quickly find $|H(j\infty)|$
  - If $m < n$: $|H(j\infty)| = 0$
  - If $m = n$: $|H(j\infty)| = a_m/b_n$
- However, $|H(0)|$ not as easily found

  $|H(0)| = \left(\frac{a_m}{b_n}\right)\left(\frac{z_1 z_2 \ldots z_m}{\omega_1 \omega_2 \ldots \omega_n}\right)$

Real Valued Poles/Zeros

- Alternatively, $H(s)$ can be written as

$$H(s) = k_{dc} \times \frac{(1 + s/z_1)(1 + s/z_2)...(1 + s/z_m)}{(1 + s/\omega_1)(1 + s/\omega_2)...(1 + s/\omega_n)}$$

- From above, can quickly find $|H(0)|$
  - $|H(0)| = k_{dc}$
- However, $|H(j\infty)|$ may be easy or difficult
  - If $m < n$: $|H(j\infty)| = 0$
  - If $m = n$: $|H(j\infty)| = k_{dc} \left(\frac{\omega_1 \omega_2 \ldots \omega_n}{z_1 z_2 \ldots z_m} \right)$
- The above root form is commonly used due to it quickly showing the dc gain value

Topics Covered

- LTI systems
- Impulse/freq response and transfer-function, $H(s)$
- Complex numbers
- Polynomial/root form for $H(s)$
- Poles/zeros and stability
- Real valued poles/zeros