Single Time-Constant Circuits

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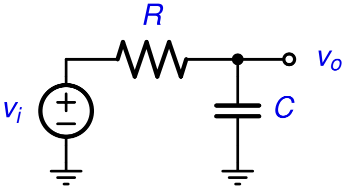
Single-Time Constant (STC) Circuits

- Here we will only concern ourselves with RC circuits
- STC circuits are circuits that when the independent sources are set to zero, the circuit can be reduced to a single capacitor and single resistor
- STC circuits have a first-order $H(s)$
- 2 main types of STC circuits
  - Lowpass
  - Highpass
- The important parameter is τ
  - τ determines the speed of settling
  - $\tau = RC$ where R and C are the single capacitor/resistor that occur when independent sources are set to zero and the circuit is reduced.
  - In the freq domain, $\omega_o = 1/\tau$
  - ω_o is the pole frequency (3 dB freq)

Lowpass STC Circuit (Freq Response)



$$H(s) \equiv \frac{V_o(s)}{V_i(s)} = \frac{1/sC}{(1/sC) + R} = \frac{1}{1 + sRC} \tag{1}$$

- Defining $\tau \equiv RC$ and $\omega_o \equiv 1/\tau = 1/(RC)$

$$H(s) = \frac{1}{1 + s\tau} = \frac{1}{1 + (s/\omega_o)} \tag{2}$$

Lowpass STC Circuit (Freq Response)

- In general, a lowpass STC transfer-function is ...

$$H(s) = \frac{K_{dc}}{1 + (s/\omega_o)} \tag{3}$$

  - where $\omega_o = 1/\tau$

- Magnitude response

$$|H(j\omega)| = \left| \frac{K_{dc}}{1 + (j\omega/\omega_o)} \right| = \left| \frac{K_{dc}}{\sqrt{1 + (\omega/\omega_o)^2}} \right| \tag{4}$$

  - For ...

$$\begin{aligned} \omega \ll \omega_o \quad &\Rightarrow \quad |H(j\omega)| \approx |K_{dc}| \\ \omega = \omega_o \quad &\Rightarrow \quad |H(j\omega)| = \tfrac{|K_{dc}|}{\sqrt{2}} \qquad \text{(3dB freq)} \\ \omega \gg \omega_o \quad &\Rightarrow \quad |H(j\omega)| = \tfrac{|K_{dc}|\omega_o}{\omega} \end{aligned}$$

Lowpass STC Circuit (Freq Response)

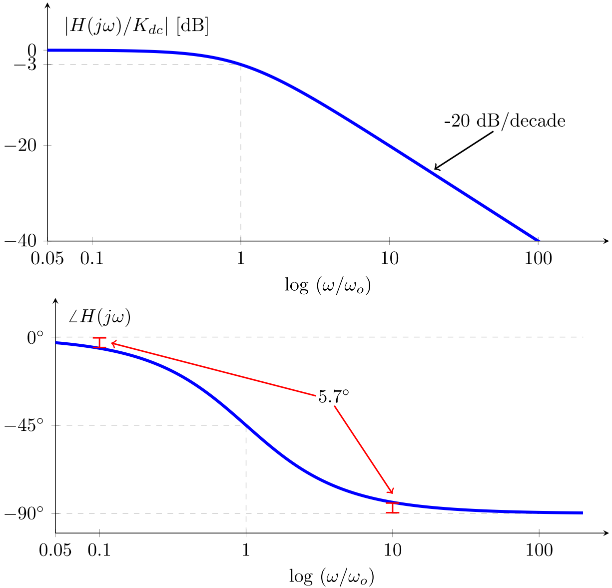
- Phase response (assume $K_{dc} > 0$)

$$\angle H(j\omega) = \angle(K_{dc}) - \angle(1 + (j\omega/\omega_o)) = -\tan^{-1}(\omega/\omega_o) \qquad (5)$$

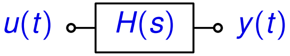
  – For ...

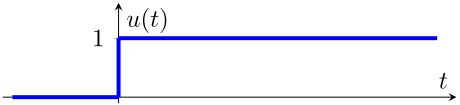
$$\begin{aligned} \omega \ll \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx 0^\circ \\ \omega = \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx -45^\circ \\ \omega \gg \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx -90^\circ \end{aligned}$$

Lowpass STC Circuit (Freq Response)



STC Circuit - Time (Step Response)





- Step response
  - For **both lowpass and highpass** STC circuits, the step response settles in an exponential behavior

$$y(t) = Y_\infty - (Y_\infty - Y_{0+})e^{-t/\tau} \tag{6}$$

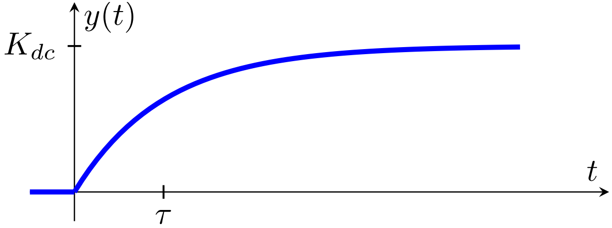
  - Y_∞ is the final value
  - Y_{0+} is the initial value
  - τ is the time-constant

Lowpass STC Circuit - Time (Step Response)

$$H(s) = \frac{K_{dc}}{1 + (s/\omega_o)} \qquad \omega_o = 1/\tau \tag{7}$$

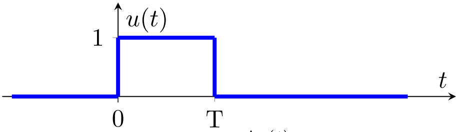
- $Y_\infty = K_{dc}; \quad Y_{0+} = 0$

$$y(t) = K_{dc}(1 - e^{-t/\tau}) \tag{8}$$



- Initial slope at $t = 0$ is K_{dc}/τ

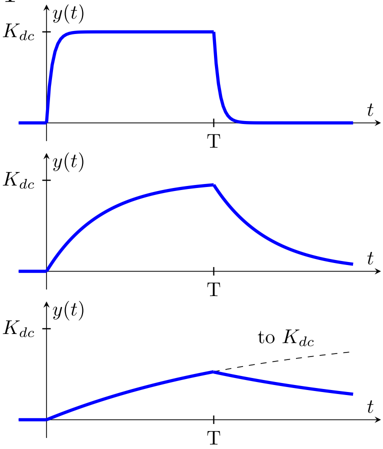
Lowpass STC Circuit - Time (Pulse Response)



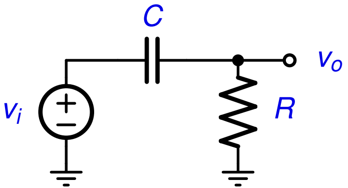
$\tau \ll T$

$\tau = T/3$

$\tau \gg T$



Highpass STC Circuit (Freq Response)



$$H(s) \equiv \frac{V_o(s)}{V_i(s)} = \frac{R}{(1/sC) + R} = \frac{sCR}{sCR + 1} = \frac{s}{s + (1/RC)} \tag{9}$$

- Defining $\tau \equiv RC$ and $\omega_o \equiv 1/\tau = 1/(RC)$

$$H(s) = \frac{s}{s + (1/\tau)} = \frac{s}{s + \omega_o} \tag{10}$$

Highpass STC Circuit (Freq Response)

- In general, a highpass STC transfer-function is ...

$$H(s) = \frac{K_\infty s}{s + \omega_o} \tag{11}$$

  – where $\omega_o = 1/\tau$

- Magnitude response

$$|H(j\omega)| = \left|\frac{K_\infty \omega}{j\omega + \omega_o}\right| = \left|\frac{K_\infty}{\sqrt{1 + (\omega_o/\omega)^2}}\right| \tag{12}$$

  – For ...

$$\begin{array}{lcll} \omega \ll \omega_o & \Rightarrow & |H(j\omega)| \approx \frac{|K_\infty|\omega}{\omega_o} & \\ \omega = \omega_o & \Rightarrow & |H(j\omega)| = \frac{|K_\infty|}{\sqrt{2}} & \text{(3dB freq)} \\ \omega \gg \omega_o & \Rightarrow & |H(j\omega)| = |K_\infty| & \end{array}$$

Highpass STC Circuit (Freq Response)

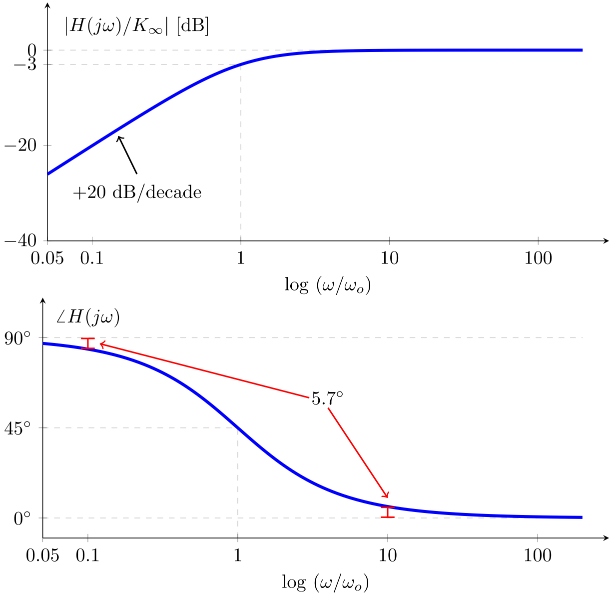
- Phase response (assume $K_\infty > 0$)

$$\angle H(j\omega) = \angle(jK_\infty \omega) - \angle(\omega_o + j\omega) = 90^\circ - \tan^{-1}(\omega/\omega_o) \qquad (13)$$

  – For ...

$$\begin{aligned} \omega \ll \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx 90^\circ \\ \omega = \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx 45^\circ \\ \omega \gg \omega_o \quad &\Rightarrow \quad \angle H(j\omega) \approx 0^\circ \end{aligned}$$

Highpass STC Circuit (Freq Response)



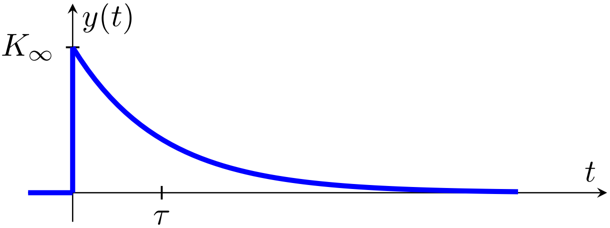
Highpass STC Circuit - Time (Step Response)

$$H(s) = \frac{K_\infty s}{s + \omega_o} \qquad \omega_o = 1/\tau \tag{14}$$

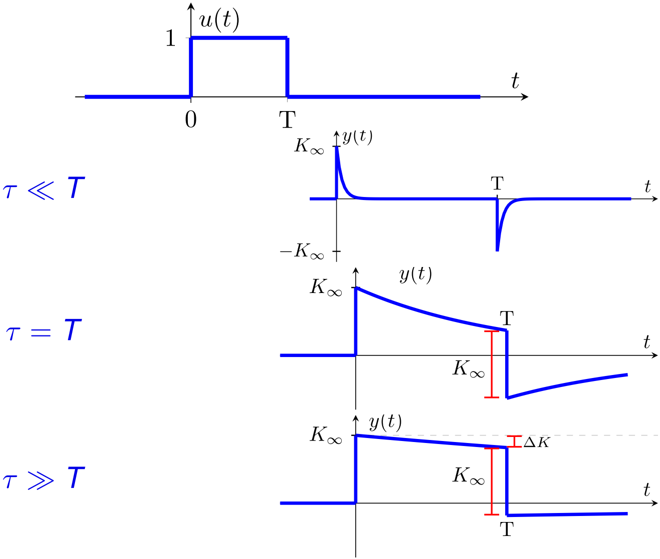
$$y(t) = Y_\infty - (Y_\infty - Y_{0+})e^{-t/\tau}$$

- $Y_\infty = 0; \quad Y_{0+} = K_\infty$

$$y(t) = K_\infty e^{-t/\tau} \tag{15}$$



- Initial slope at $t = 0$ is $-K_\infty/\tau$



u(t)
1
0
T
t
τ ≪ T
τ = T
τ ≫ T
y(t)
K∞
−K∞
ΔK

Highpass STC Circuit - Time (Pulse Response)

- For the case $\tau \gg T$
  - ΔK can be found using the exponential equation
  - However, it can also be approximately found since the decay is mostly in the linear region at the beginning
  - Slope is $-K_\infty/\tau$ resulting in

$$\Delta K \approx \frac{K_\infty}{\tau} \times T \tag{16}$$

  - So the percentage ”sag”, SAG, is given by

$$\mathrm{SAG} = \frac{\Delta K}{K_\infty} \times 100 = \frac{T}{\tau} \times 100 \tag{17}$$

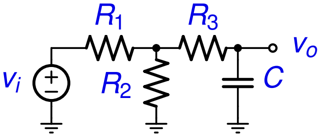
  - So the SAG decreases as τ is increased

Finding τ from STC Circuits

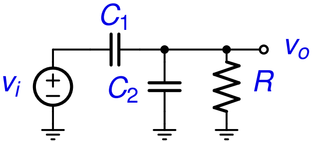
- STC circuits are circuits that when the independent sources are set to zero, the circuit can be reduced to a single capacitor and single resistor.
- The following are STC circuits
  - Any circuit with a single capacitor and multiple resistors
  - Any circuit with a single resistor and multiple capacitors
- To find τ
  - Set all independent sources to zero
  - If a single capacitor... find the equivalent resistance, R_{eq} that is across the capacitor... $\tau = R_{eq}C$
  - If a single resistor... find the equivalent capacitance, C_{eq} that is across the resistor... $\tau = RC_{eq}$

Finding τ from STC Circuits

- Examples

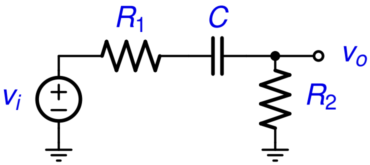


- When $v_i = 0$, the source becomes a short circuit
- $R_{eq} = (R_1 || R_2) + R_3$
- $\tau = R_{eq} C$

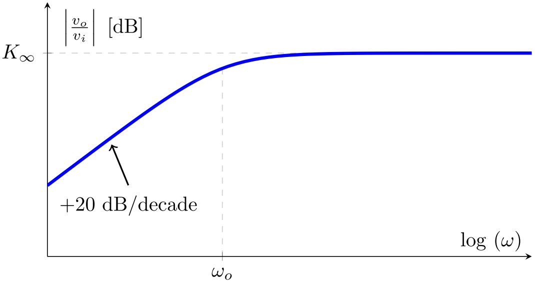


- When $v_i = 0$, the source becomes a short circuit
- $C_{eq} = C_1 + C_2$
- $\tau = RC_{eq}$

Finding τ from STC Circuits



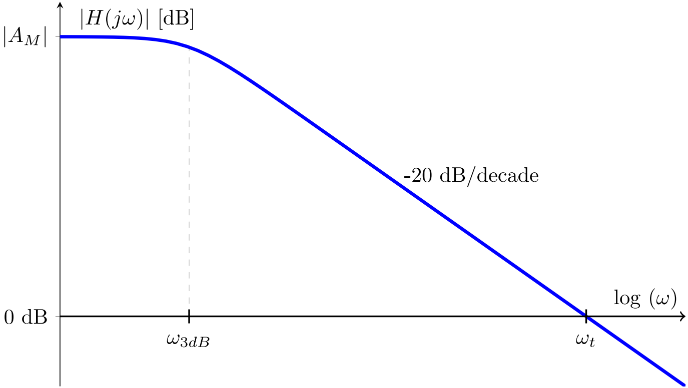
- $\tau = C(R_1 + R_2)$ $\qquad \omega_o = \frac{1}{C(R_1+R_2)}$
- $K_\infty = \left|\frac{v_o}{v_i}(j\infty)\right| = \frac{R_2}{R_1+R_2}$ since the cap is a short circuit at high freq



Unity-Gain Freq of Lowpass STC

- Consider a lowpass STC circuit with a LARGE dc gain

$$H(s) = \frac{A_M}{1 + (s/\omega_{3dB})} \qquad |A_M| \gg 1 \tag{18}$$



Unity-Gain Freq of Lowpass STC

- ω_t occurs when $|H(j\omega_t)| = 1$

$$\left| \frac{A_M}{1 + (j\omega_t/\omega_{3db})} \right| = 1 \qquad (19)$$

- since $|A_M| \gg 1$, we know $\omega_t \gg \omega_{3dB}$

$$\left| \frac{A_M}{j\omega_t/\omega_{3db}} \right| \approx 1 \qquad (20)$$

$$\omega_t \approx |A_M|\omega_{3dB} \qquad (21)$$

and in Hz

$$f_t \approx |A_M| f_{3dB} \qquad (22)$$

Topics Covered

- Lowpass STC freq and time responses
- Highpass STC freq and time responses
- Finding τ from STC circuits
- Unity gain freq of STC when large dc gain