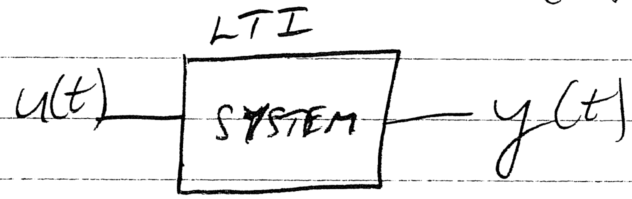
FRI

FREQUENCY RESPONSE REVIEW

CAN DEFINE A FREQUENCY RESPONSE FOR A <u>LINEAR</u> <u>TIME-INVARIANT</u> SYSTEM (LTI)



<u>LINEAR</u> IF $u_1(t)$ RESULTS IN $y_1(t)$

$u_2(t)$ RESULTS IN $y_2(t)$

THEN LINEAR IF AND ONLY IF

$u_1(t)+u_2(t)$ RESULTS IN $y_1(t)+y_2(t)$

FOR ANY $u_1(t)$, $u_2(t)$

TIME INVARIANT

TIME INVARIANT IF & ONLY IF

IF $u_1(t)$ RESULTS IN $y_1(t)$

THEN $u_1(t-\tau)$ RESULTS IN $y_1(t-\tau)$

FOR ANY $u_1(t)$ & τ

FREQUENCY RESPONSE

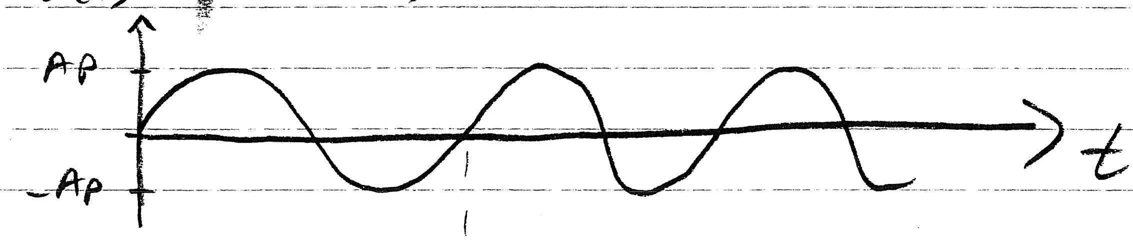
LET LTI SYSTEM HAVE TRANSFER FUNCTION $T(s)$

$$u(t) \longrightarrow \boxed{T(s)} \longrightarrow y(t)$$

FREQUENCY RESPONSE IS GIVEN BY LETTING $s = j\omega$ WHERE "ω" IS ANGULAR FREQUENCY IN RAD/S

$\omega = 2\pi f$ WHERE f IS IN $\frac{\text{CYCLES}}{\text{S}}$ OR HZ

$$u(t) = A_P \sin(\omega t) = A_P \sin(2\pi f t)$$

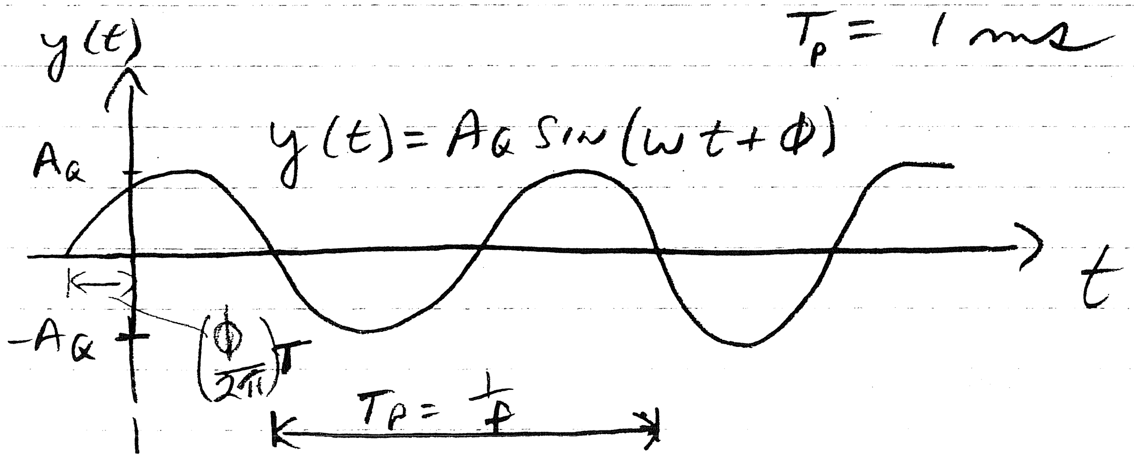


$$T_P = \frac{1}{f} = \frac{2\pi}{\omega}$$

FOR $\omega = 6.28e3$ RAD/S

$f = \omega / 2\pi = 1$ KHZ

$T_P = 1$ ms



FOR LTI

FR3

IF INPUT IS SINUSOIDAL WITH FREQ f
THEN OUTPUT IS ALSO SINUSOIDAL
WITH FREQ f BUT DIFFERENT AMPLITUDE + PHASE.

$$\frac{A_Q}{A_P} = |T(j\omega)| \quad \text{MAGNITUDE RESPONSE}$$

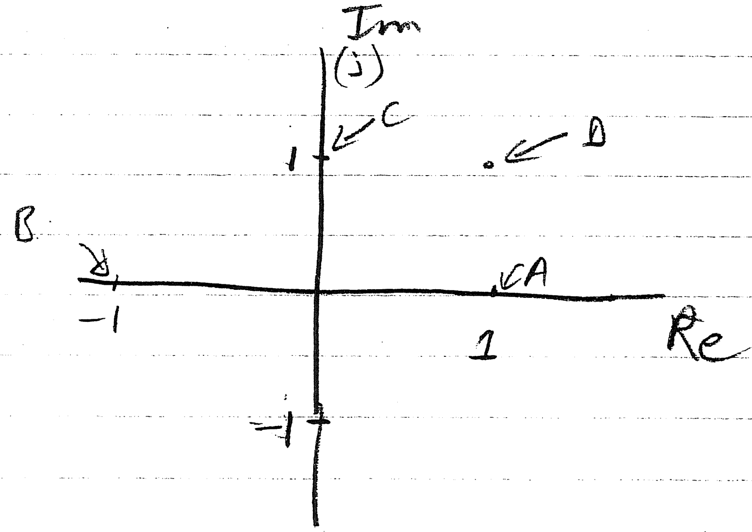
$$\phi = \angle T(j\omega)$$

$T(j\omega)$ IS A COMPLEX NUMBER WHERE
THE $|T(j\omega)|$ IS MAGNITUDE RESPONSE

+ $\angle T(j\omega)$ IS PHASE RESPONSE

FR4

COMPLEX NUMBERS



$A = 1 \qquad |A| = 1 \qquad \angle A = 0^\circ$

$B = -1 \qquad |B| = 1 \qquad \angle B = 180^\circ$

$C = j \qquad |C| = 1 \qquad \angle C = 90^\circ$

$D = 1 + j \qquad |D| = \sqrt{2} \qquad \angle D = 45^\circ$

IF $z = a + jb$ z COMPLEX
a, b REAL

$|z| = (a^2 + b^2)^{\frac{1}{2}}$

$\angle z = \tan^{-1}\left(\frac{b}{a}\right)$ IF $a > 0$

$\angle z = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ IF $a < 0$

FR4A

RATIO OF COMPLEX NUMBERS

$$Z \equiv \frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{|z_1| e^{j\angle z_1}}{|z_2| e^{j\angle z_2}}$$

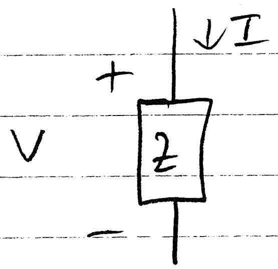
$$|Z| = \frac{|z_1|}{|z_2|} = \left(\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}\right)^{\frac{1}{2}}$$

$$\angle Z = \angle z_1 - \angle z_2$$

$$= \tan^{-1}\left(\frac{b_1}{a_1}\right) - \tan^{-1}\left(\frac{b_2}{a_2}\right) \quad \text{if } a_1 > 0,\ a_2 > 0$$

FR5

OHM'S LAW (IMPEDANCE)



$$I = \frac{V}{Z}$$

RESISTOR OF SIZE R $\Rightarrow$ $Z = R$

CAPACITOR OF SIZE C $\Rightarrow$ $Z = \frac{1}{sC}$

INDUCTOR OF SIZE L $\Rightarrow$ $Z = sL$

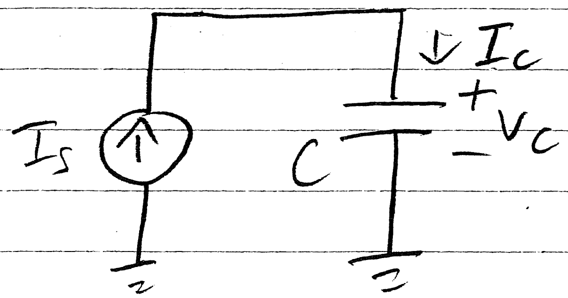
"s" IS LAPLACE TRANSFORM VARIABLE

& WE LET $s = j\omega$ TO EVALUATE WHAT HAPPENS AT FREQUENCY ω (OMEGA)

WHY IS THERE THE COMPLEX VARIABLE "j" ??

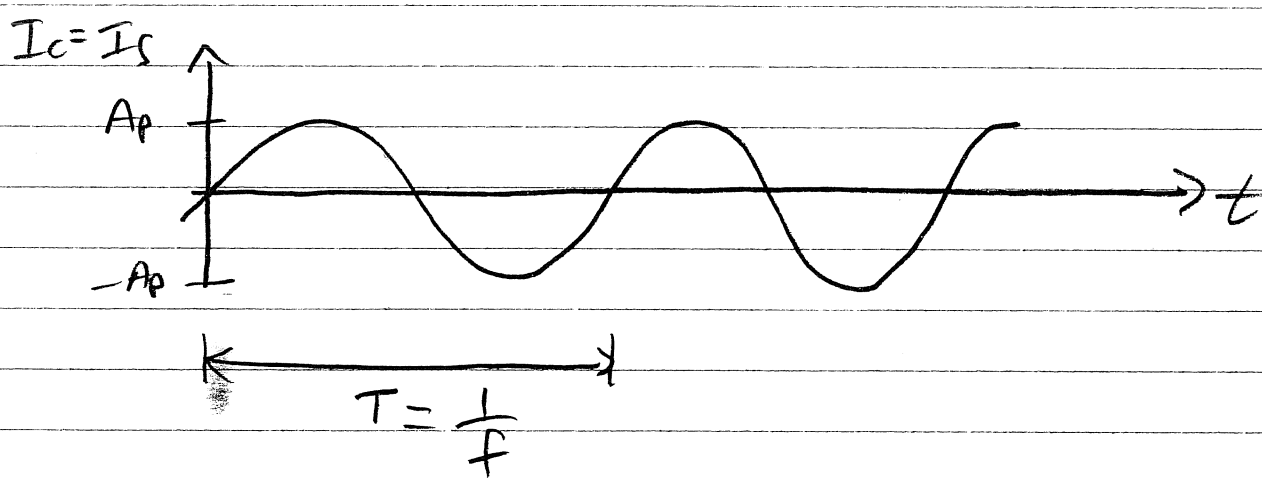
FR6

CAPACITOR



$$I_C = I_S$$

$$I_S = A_p \sin(\omega t) = A_p \sin(2\pi f t)$$



EXAMPLE

$A_p = 1\,mA$

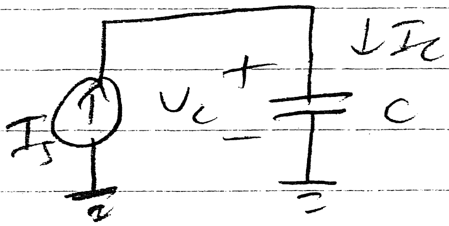
$\omega = 6.28e3$ RAD/S

$f = \omega / 2\pi = 1\,KHZ$

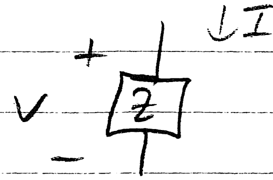
$T = 1\,ms$

IT TURNS OUT THAT VOLTAGE V_C IS 90° OUT OF PHASE WITH I_C

HENCE THE NEED FOR "j"



RECALL



$I = \frac{V}{Z}$

IN FREQ DOMAIN, TRANSFER FUNCTION CAN BE FOUND

$$T(s) \equiv \frac{V_C(s)}{I_C(s)} \Rightarrow I_C(s) = \frac{V_C(s)}{\left(\frac{1}{sC}\right)}$$

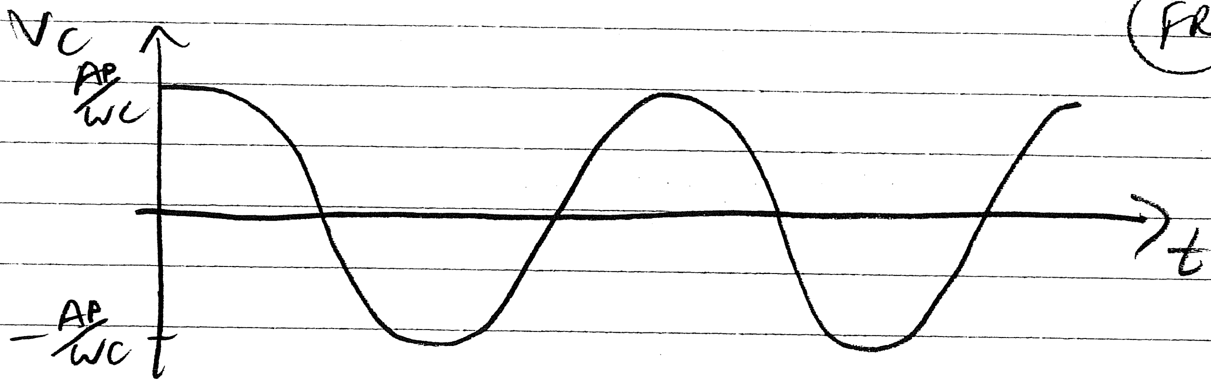
$$T(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$$

MAGNITUDE RESPONSE $\quad |T(j\omega)| = \left|\frac{1}{j\omega C}\right| = \underline{\underline{\frac{1}{\omega C}}}$

PHASE RESPONSE $\quad \angle T(j\omega) = \angle\left(\frac{1}{j\omega C}\right)$

$$\angle T(j\omega) = \angle(1) - \angle(j\omega C)$$
$$= 0^\circ - 90^\circ = \underline{\underline{-90^\circ}}$$

FR8



IF $C = 1\mu F$

$$\frac{A_P}{\omega C} = \frac{1\,mA}{(6.28e3)(1\,\mu F)} = 0.159\ V$$

FR9

TRANSFER-FUNCTION OF LTI

- RESTRICT OURSELVES TO
  - REAL-VALUED IMPULSE RESPONSE
  - CIRCUITS WITH LUMPED ELEMENTS
    - RESISTORS, CAPACITORS, INDUCTORS, INDEPENDENT + DEPENDENT VOLTAGE + CURRENT SOURCES

$$T(s) = \frac{a_0 + a_1 s + a_2 s^2 + \dots + a_m s^m}{1 + b_1 s + b_2 s^2 + \dots + b_n s^n} \quad (1)$$

OR

$$T(s) = \left(\frac{a_m}{b_n}\right) \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+w_1)(s+w_2)\cdots(s+w_n)} \quad (2)$$

FOR STABILITY $n \geq m$

(1) POLYNOMIAL FORM

(2) ROOT FORM SHOWS ZEROS $-z_i$

POLES $-w_i$

IN GENERAL, z_i & w_i CAN BE COMPLEX VALUES + OCCUR IN COMPLEX CONJUGATE PAIRS

HOWEVER, FOR THIS FREQ ANALYSIS SECTION ALL z_i + w_i REAL VALUED

FR10

IN OTHER WORDS, ALL POLES & ZEROS WILL BE ON THE REAL-AXIS

ALSO FOR A STABLE SYSTEM, ALL POLES WILL BE IN NEGATIVE HALF PLANE (AND HENCE ON NEGATIVE REAL-AXIS HERE)

ZEROS WHERE $T(s) = 0$

ZERO AT $s = z_1 \Rightarrow T(z_1) = 0$

POLES WHERE $T(s) \to \infty$

POLE AT $s = w_i \Rightarrow T(w_i) \to \infty$

EX $T(s) = \dfrac{1}{s+2}$ NO ZEROS

POLE $w_1 = -2$

EX $T(s) = \dfrac{s}{s+3}$ ZERO $z_1 = 0$

POLE $w_1 = -3$

$T(s) = \dfrac{s(s-2)}{(s+1)(s+3)}$ ZEROS: $z_1 = 0$ $z_2 = 2$

POLES: $p_1 = -1$ $p_2 = -3$

FR11

$$T(s) = \left(\frac{a_m}{b_n}\right) \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+w_1)(s+w_2)\cdots(s+w_n)}$$

STRICTLY SPEAKING POLES AT $-w_i$
AND ZEROS AT $-z_i$

HOWEVER (FOR REAL AXIS POLES/ZEROS)
OFTEN SAY POLE FREQUENCY AT w_i
OR ZERO FREQUENCY AT z_i
SINCE IF WE ONLY CONSIDER
POLE (OR ZERO) AT w_i

$T(s)$ IS REDUCED (INCREASED) BY $\frac{1}{\sqrt{2}}$ AT $T(jw_i)$

EX

$$T(s) = \frac{1}{s+1}$$

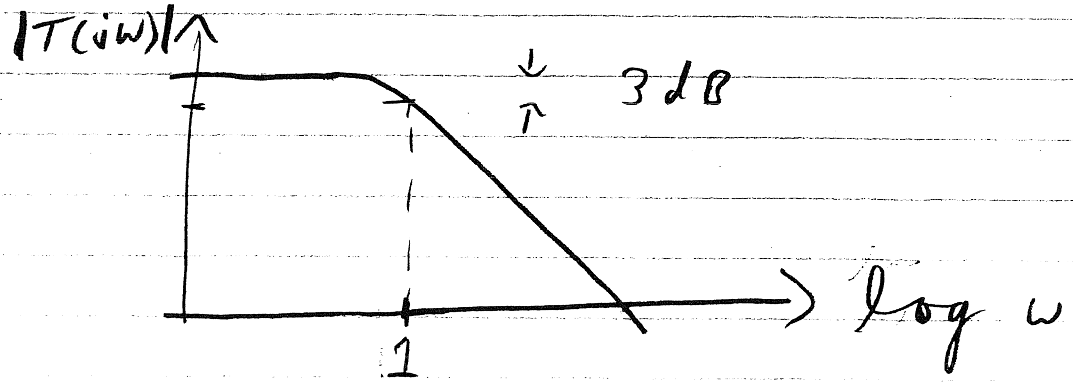
$$T(0) = 1$$

$$T(j\infty) = 0$$

$$T(j1) = \frac{1}{\sqrt{2}}$$

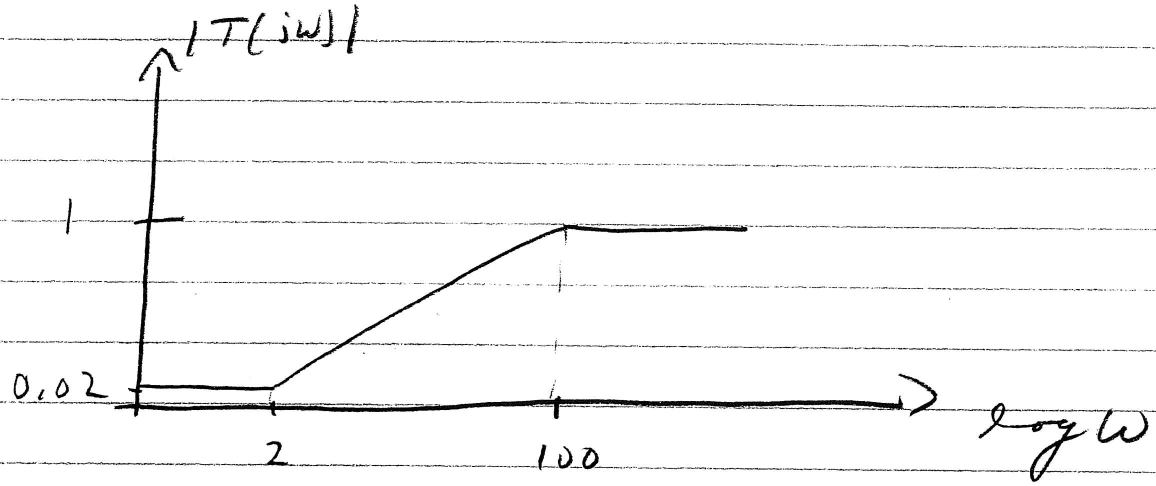
SO FOR $w = 1$ RAD/S

$|T(j1)| = \frac{1}{\sqrt{2}} T(0)$ OR 3dB DOWN



EX $\quad T(s) = \dfrac{s+2}{s+100}$

$T(0) = 0.02 \qquad T(j\infty) = 1$



$|T(j2)| \approx \sqrt{2}\,|T(0)|$ 3dB INCREASE

$|T(j100)| \approx \dfrac{1}{\sqrt{2}}\,|T(j\infty)|$ 3dB DECREASE

ZERO AT 2 RAD/S

POLE AT 100 RAD/S

FR13

ALTERNATE ROOT FORM

$$T(s) = \left(\frac{a_m}{b_n}\right) \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+w_1)(s+w_2)\cdots(s+w_n)}$$

HERE

$$T(j\infty) = \begin{cases} \dfrac{a_m}{b_n} & \text{IF } m = n \\ 0 & \text{IF } m < n \end{cases}$$

$$T(0) = \left(\frac{a_m}{b_n}\right) \frac{z_1 z_2 \cdots z_m}{w_1 w_2 \cdots w_n}$$

ALTERNATIVELY $T(s)$ WRITTEN AS

$$T(s) = k_{dc} \frac{(1+s/z_1)(1+s/z_2)\cdots(1+s/z_m)}{(1+s/w_1)(1+s/w_2)\cdots(1+s/w_n)}$$

$T(j\infty) = 0$ IF $m < n$

(COMPLICATED IF $m = n$)

$T(0) = k_{dc}$