

# Oversampling Converters



## Motivation

- Popular approach for medium-to-low speed A/D and D/A applications requiring high resolution

### Easier Analog

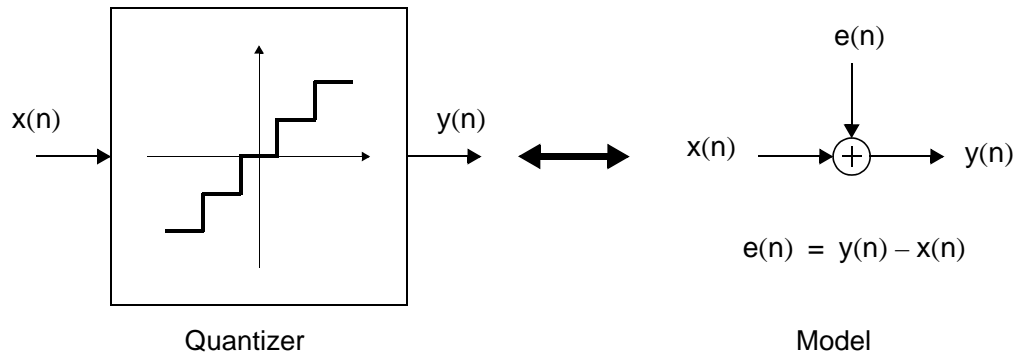
- reduced matching tolerances
- relaxed anti-aliasing specs
- relaxed smoothing filters

### More Digital Signal Processing

- Needs to perform strict anti-aliasing or smoothing filtering
- Also removes shaped quantization noise and decimation (or interpolation)



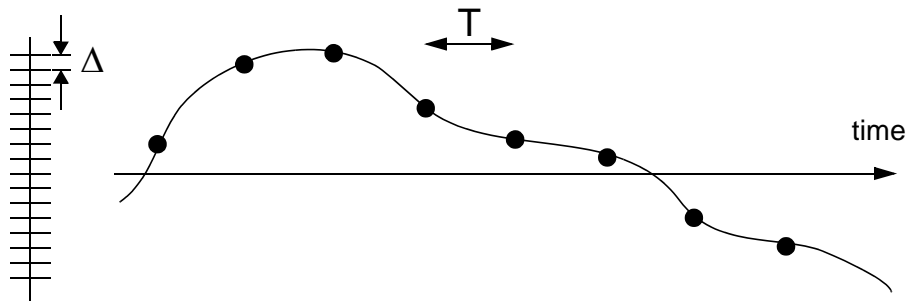
## Quantization Noise



- Above model is exact
  - approx made when assumptions made about  $e(n)$
- Often assume  $e(n)$  is white, uniformly distributed number between  $\pm\Delta/2$
- $\Delta$  is difference between two quantization levels



## Quantization Noise

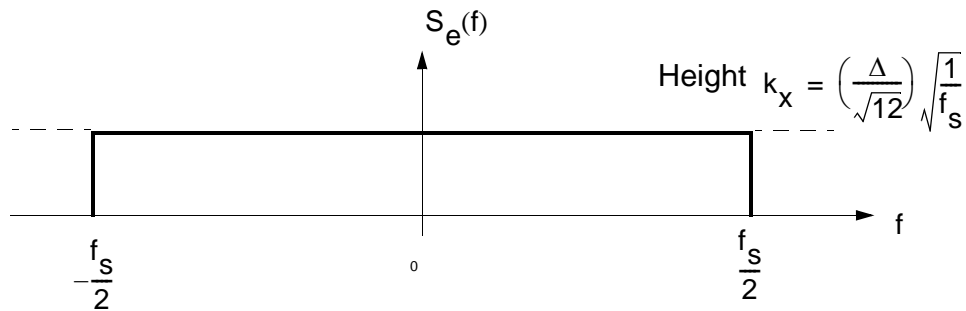


- White noise assumption reasonable when:
  - fine quantization levels
  - signal crosses through many levels between samples
  - sampling rate not synchronized to signal frequency
- Sample lands somewhere in quantization interval leading to random error of  $\pm\Delta/2$



## Quantization Noise

- Quantization noise power shown to be  $\Delta^2/12$  and is ***independent of sampling frequency***
- If white, then spectral density of noise,  $S_e(f)$ , is constant.

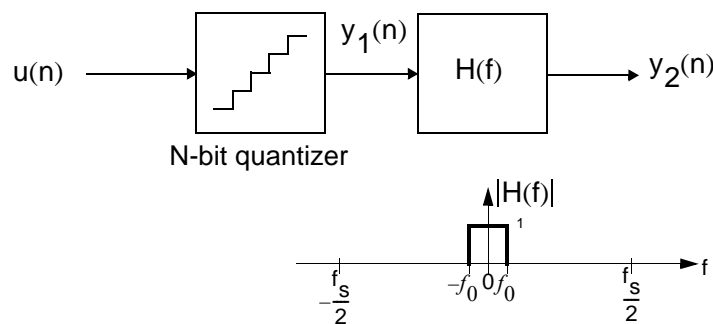


## Oversampling Advantage

- Oversampling occurs when signal of interest is bandlimited to  $f_0$  but we sample higher than  $2f_0$
- Define oversampling-rate

$$\text{OSR} = f_s / (2f_0) \tag{1}$$

- After quantizing input signal, pass it through a brickwall digital filter with passband up to  $f_0$



## Oversampling Advantage

- Output quantization noise after filtering is:

$$P_e = \int_{-f_s/2}^{f_s/2} S_e^2(f) |H(f)|^2 df = \int_{-f_0}^{f_0} k_x^2 df = \frac{\Delta^2}{12} \left( \frac{1}{OSR} \right) \quad (2)$$

- Doubling OSR reduces quantation noise power by 3dB (i.e. 0.5 bits/octave)
- Assuming peak input is a sinusoidal wave with a peak value of  $2^N (\Delta/2)$  leading to

$$P_s = ((\Delta 2^N) / (2\sqrt{2}))^2$$

- Can also find peak SNR as:

$$SNR_{max} = 10 \log \left( \frac{P_s}{P_e} \right) = 10 \log \left( \frac{3}{2} 2^{2N} \right) + 10 \log(OSR) \quad (3)$$



## Oversampling Advantage

### Example

- A dc signal with 1V is combined with a noise signal uniformly distributed between  $\pm\sqrt{3}$  giving 0 dB SNR. — {0.94, -0.52, -0.73, 2.15, 1.91, 1.33, -0.31, 2.33}.
- Average of 8 samples results in 0.8875
- Signal adds linearly while noise values add in a square-root fashion — noise filtered out.

### Example

- 1-bit A/D gives 6dB SNR.
- To obtain 96dB SNR requires 30 octaves of oversampling ( (96-6)/3 dB/octave )
- If  $f_0 = 25$  kHz,  $f_s = 2^{30} \times f_0 = 54,000$  GHz !



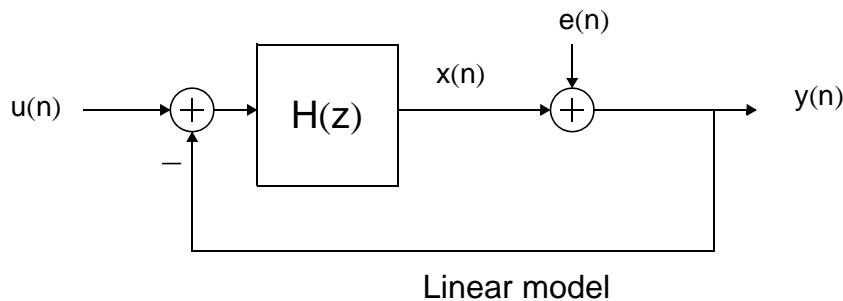
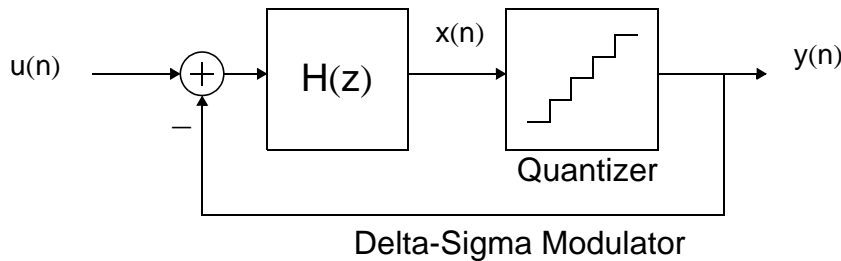
## Advantage of 1-bit D/A Converters

- Oversampling improves SNR but not linearity
- To achieve 16-bit linear converter using a 12-bit converter, 12-bit converter must be linear to 16 bits
  - i.e. integral nonlinearity better than  $1/2^4$  LSB
- A 1-bit D/A is ***inherently linear***
  - 1-bit D/A has only 2 output points
  - 2 points always lie on a straight line
- Can achieve better than 20 bits linearity without trimming (will likely have gain and offset error)
- Second-order effects (such as D/A memory or signal-dependent reference voltages) will limit linearity.



## Oversampling with Noise Shaping

- Place the quantizer in a feedback loop



## Oversampling with Noise Shaping

- Shapes quantization noise away from signal band of interest

### Signal and Noise Transfer-Functions

$$S_{TF}(z) \equiv \frac{Y(z)}{U(z)} = \frac{H(z)}{1 + H(z)} \quad (4)$$

$$N_{TF}(z) \equiv \frac{Y(z)}{E(z)} = \frac{1}{1 + H(z)} \quad (5)$$

$$Y(z) = S_{TF}(z)U(z) + N_{TF}(z)E(z) \quad (6)$$

- Choose  $H(z)$  to be large over 0 to  $f_0$
- Resulting quantization noise near 0 where  $H(z)$  large
- Signal transfer-function near 1 where  $H(z)$  large



## Oversampling with Noise Shaping

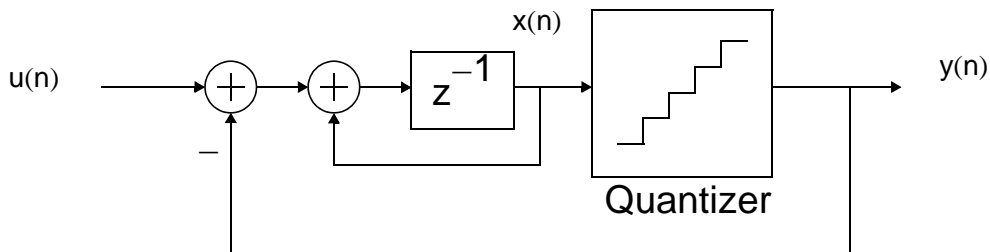
- Input signal is limited to range of quantizer output when  $H(z)$  large
- For 1-bit quantizers, input often limited to 1/4 quantizer outputs
- Out-of-band signals can be larger when  $H(z)$  small
- Stability of modulator can be an issue (particularly for higher-orders of  $H(z)$ )
- Stability defined as when input to quantizer becomes so large that quantization error greater than  $\pm\Delta/2$  — said to “overload the quantizer”



## First-Order Noise Shaping

- Choose  $H(z)$  to be a discrete-time integrator

$$H(z) = \frac{1}{z-1} \quad (7)$$



- If stable, average input of integrator must be zero
- Average value of  $u(n)$  must equal average of  $y(n)$



## Example

- The output sequence and state values when a dc input,  $u(n)$ , of  $1/3$  is applied to a 1'st order modulator with a two-level quantizer of  $\pm 1.0$ . Initial state for  $x(n)$  is 0.1.

n	$x(n)$	$x(n+1)$	$y(n)$	$e(n)$
0	0.1	-0.5667	1.0	0.9
1	-0.5667	0.7667	-1.0	-0.4333
2	0.7667	0.1	1.0	0.2333
3	0.1	-0.5667	1.0	0.9
4	-0.5667	0.7667	-1.0	-0.4333
5	...	...	...	...

- Average of  $y(n)$  is  $1/3$  as expected
- Periodic quantization noise in this case



## Transfer-Functions

### Signal and Noise Transfer-Functions

$$S_{TF}(z) = \frac{Y(z)}{U(z)} = \frac{1/(z-1)}{1+1/(z-1)} = z^{-1} \quad (8)$$

$$N_{TF}(z) = \frac{Y(z)}{E(z)} = \frac{1}{1+1/(z-1)} = (1-z^{-1}) \quad (9)$$

- Noise transfer-function is a discrete-time differentiator (i.e. a highpass filter)

$$\begin{aligned} N_{TF}(f) &= 1 - e^{-j2\pi f/f_s} = \frac{e^{j\pi f/f_s} - e^{-j\pi f/f_s}}{2j} \times 2j \times e^{-j\pi f/f_s} \\ &= \sin\left(\frac{\pi f}{f_s}\right) \times 2j \times e^{-j\pi f/f_s} \end{aligned} \quad (10)$$



## Signal to Noise Ratio

### Magnitude of noise transfer-function

$$|N_{TF}(f)| = 2 \sin\left(\frac{\pi f}{f_s}\right) \quad (11)$$

### Quantization noise power

$$P_e = \int_{-f_0}^{f_0} S_e^2(f) |N_{TF}(f)|^2 df = \int_{-f_0}^{f_0} \left(\frac{\Delta^2}{12}\right) \frac{1}{f_s} \left[2 \sin\left(\frac{\pi f}{f_s}\right)\right]^2 df \quad (12)$$

- Assuming  $f_0 \ll f_s$  (i.e.,  $OSR \gg 1$ )

$$P_e \cong \left(\frac{\Delta^2}{12}\right) \left(\frac{\pi^2}{3}\right) \left(\frac{2f_0}{f_s}\right)^3 = \frac{\Delta^2 \pi^2}{36} \left(\frac{1}{OSR}\right)^3 \quad (13)$$



## Max SNR

- Assuming peak input is a sinusoidal wave with a peak value of  $2^N(\Delta/2)$  leading to

$$P_s = ((\Delta 2^N)/(2\sqrt{2}))^2$$

- Can find peak SNR as:

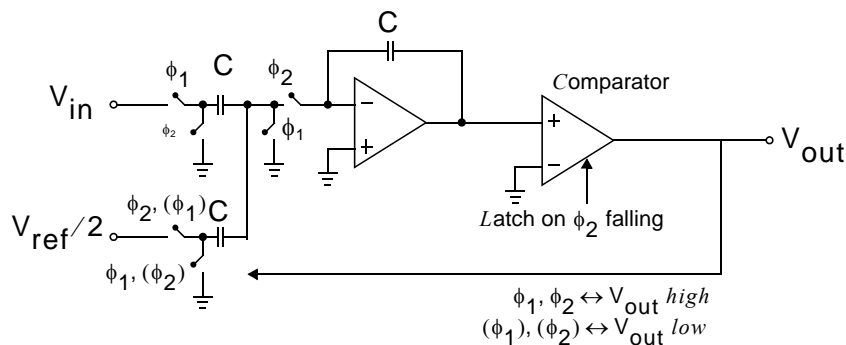
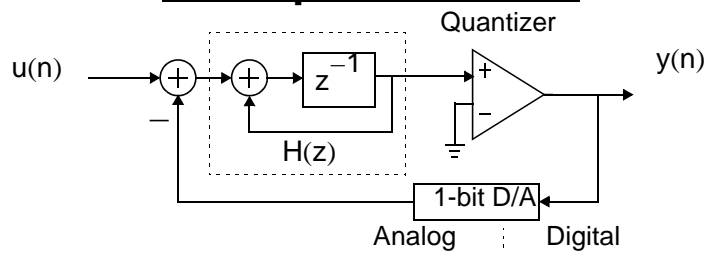
$$\begin{aligned} \text{SNR}_{max} &= 10 \log\left(\frac{P_s}{P_e}\right) \\ &= 10 \log\left(\frac{3}{2}2^{2N}\right) + 10 \log\left[\frac{3}{\pi^2}(OSR)^3\right] \end{aligned} \quad (14)$$

$$\text{SNR}_{max} = 6.02N + 1.76 - 5.17 + 30 \log(OSR) \quad (15)$$

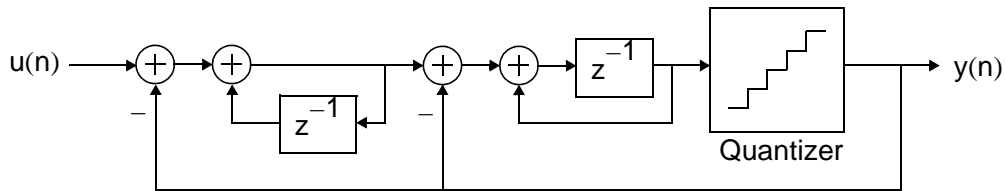
- Doubling OSR gives an SNR improvement 9 dB (a benefit of 1.5 bits/octave)



## SC Implementation



## Second-Order Noise Shaping



$$S_{TF}(f) = z^{-1} \quad (16)$$

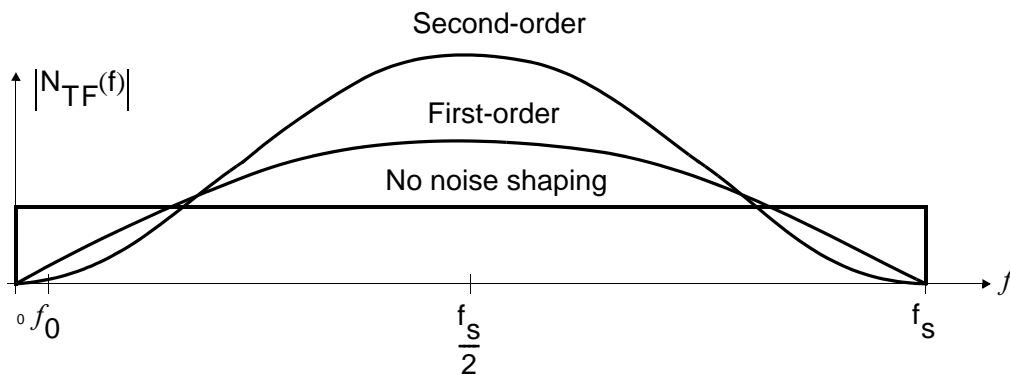
$$N_{TF}(f) = (1 - z^{-1})^2 \quad (17)$$

$$SNR_{max} = 6.02N + 1.76 - 12.9 + 50 \log(OSR) \quad (18)$$

- Doubling  $OSR$  improves SNR by 15 dB (i.e., a benefit of 2.5 bits/octave)



## Noise Transfer-Function Curves



- Out-of-band noise increases for high-order modulators
- Out-of-band noise peak controlled by poles of noise transfer-function
- Can also spread zeros over band-of-interest



## Example

- 90 dB SNR improvement from A/D with  $f_0 = 25$  kHz

### Oversampling with no noise shaping

- From before, straight oversampling requires a sampling rate of 54,000 GHz.

### First-Order Noise Shaping

- Lose 5 dB (see (15)), require 95 dB divided by 9 dB/octave, or 10.56 octaves —  $f_s = 2^{10.56} \times 2f_0 \cong 75$  MHz

### Second-Order Noise Shaping

- Lose 13 dB, required 103 dB divided by 15 dB/octave,  $f_s = 5.8$  MHz (does not account for reduced input range needed for stability).

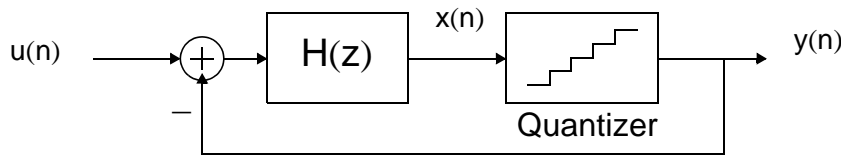


## Quantization Noise Power of 1-bit Modulators

- If output of 1-bit mod is  $\pm 1$ , total power of output signal,  $y(n)$ , is normalized power of 1 watt.
- Signal level often limited to well below  $\pm 1$  level in higher-order modulators to maintain stability
- For example, if maximum peak level is  $\pm 0.25$ , max signal power is 62.5 mW.
- Max signal is approx 12 dB below quantization noise (but most noise in different frequency region)
- Quantization filter must have dynamic range capable of handling full power of  $y(n)$  at input.
- Easy for A/D — digital filter
- More difficult for D/A — analog filter



## Zeros of NTF are poles of $H(z)$



- Write  $H(z)$  as

$$H(z) = \frac{N(z)}{D(z)} \quad (19)$$

- NTF is given by:

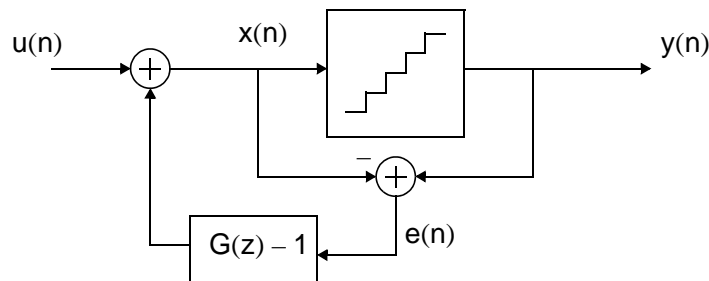
$$\text{NTF}(z) = \frac{1}{1 + H(z)} = \frac{D(z)}{D(z) + N(z)} \quad (20)$$

- If poles of  $H(z)$  are well-defined then so are zeros of NTF



## Error-Feedback Structure

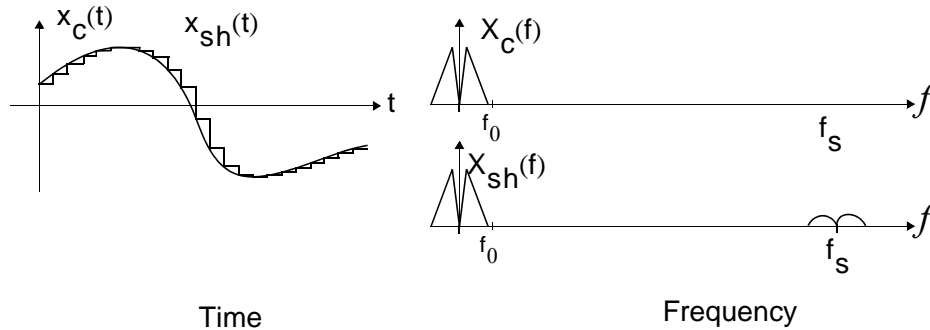
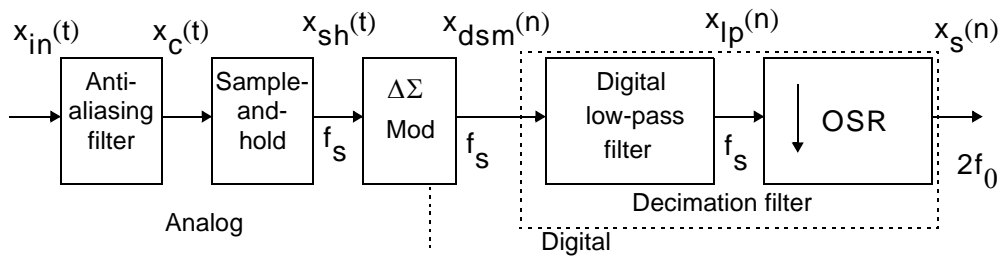
- Alternate structure to interpolative



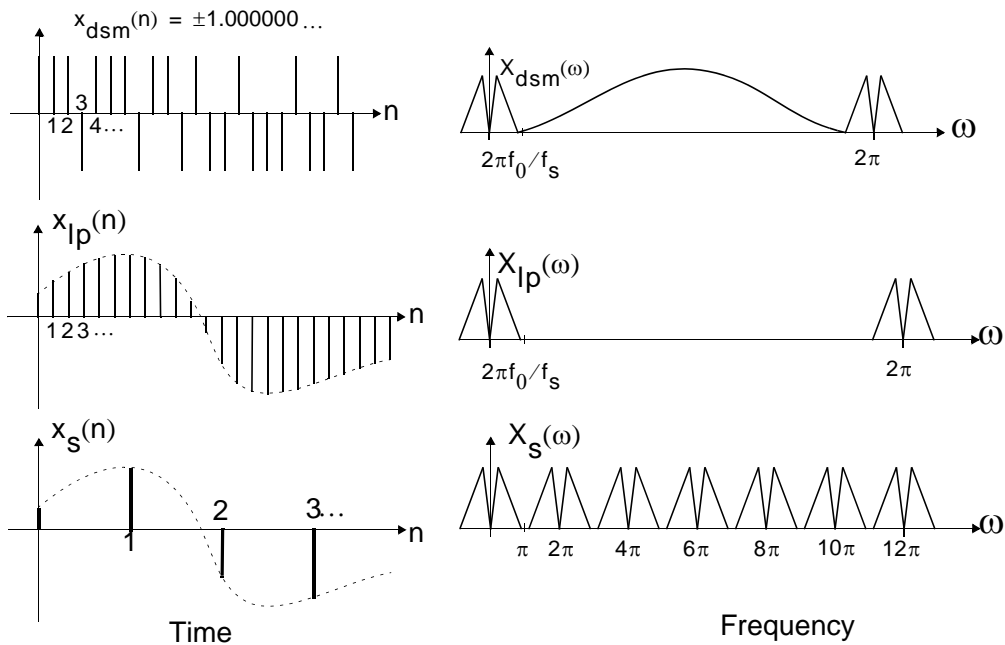
- Signal transfer-function equals unity while noise transfer-function equals  $G(z)$
- First element of  $G(z)$  equals 1 for no delay free loops
- First-order system —  $G(z) - 1 = -z^{-1}$
- More sensitive to coefficient mismatches



## Architecture of Delta-Sigma A/D Converters



## Architecture of Delta-Sigma A/D Converters

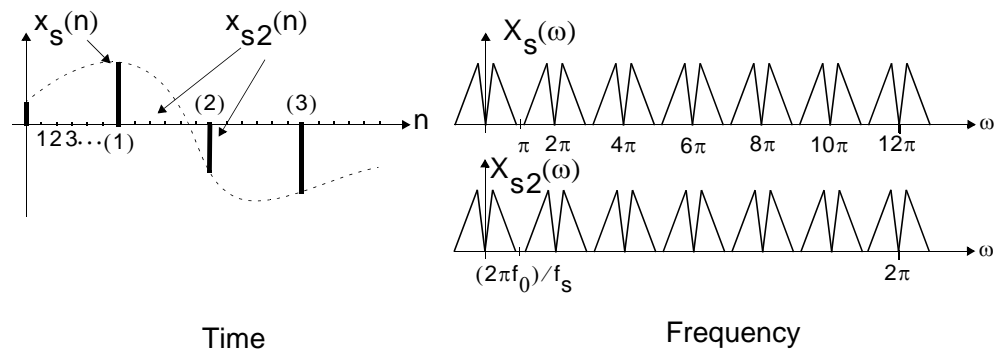
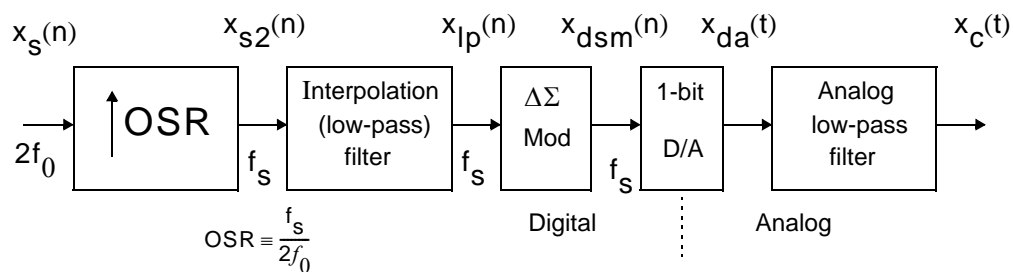


## Architecture of Delta-Sigma A/D Converters

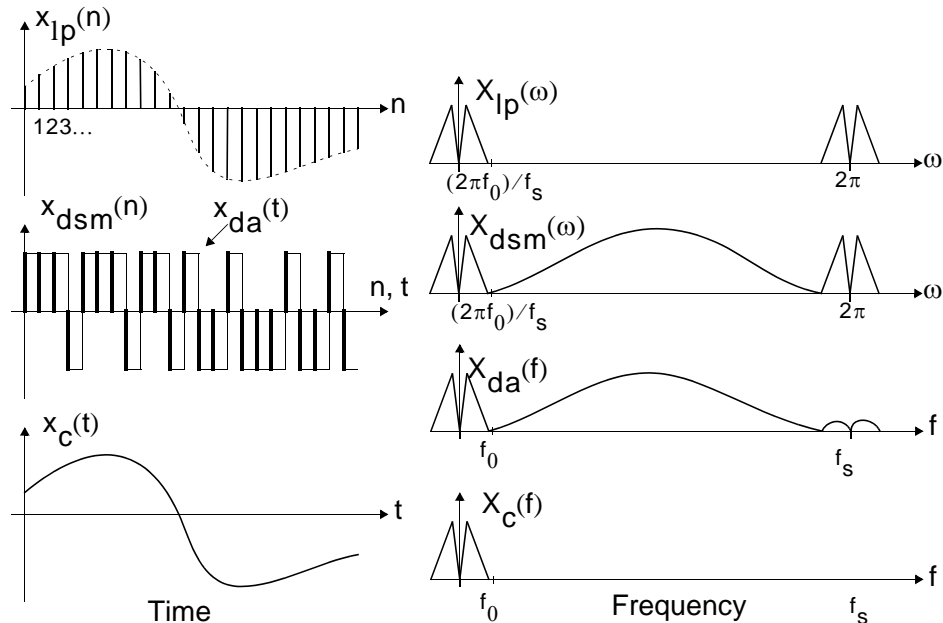
- Relaxes analog anti-aliasing filter
- Strict anti-aliasing done in digital domain
- Must also remove quantization noise before downsampling (or aliasing occurs)
- Commonly done with a multi-stage system
- Linearity of D/A in modulator important — results in overall nonlinearity
- Linearity of A/D in modulator unimportant (effects reduced by high gain in feedback of modulator)



## Architecture of Delta-Sigma D/A Converters



## Architecture of Delta-Sigma D/A Converters

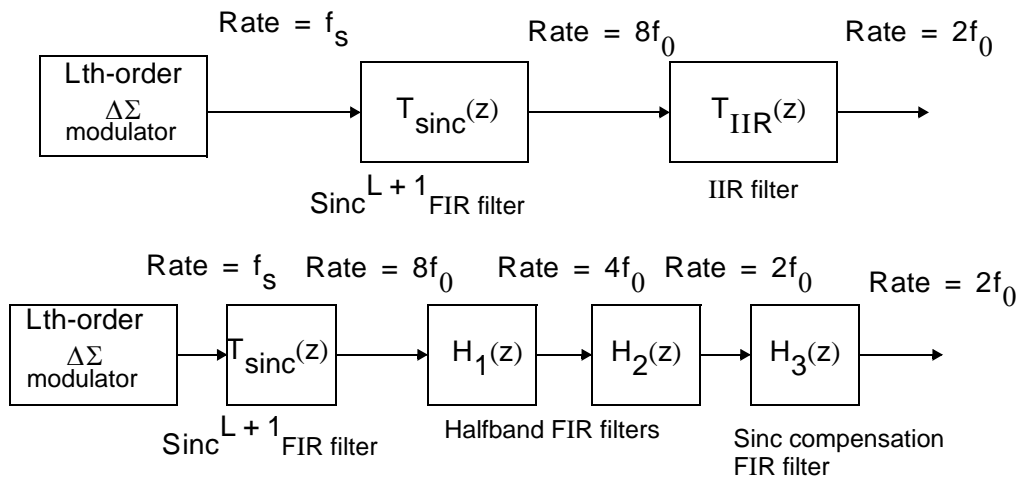


## Architecture of Delta-Sigma D/A Converters

- Relaxes analog smoothing filter (many multibit D/A converters are oversampled without noise shaping)
- Smoothing filter of first few images done in digital (then often below quantization noise)
- Order of lowpass filter should be at least one order higher than that of modulator
- Results in noise dropping off (rather than flat)
- Analog filter must attenuate quantization noise and should not modulate noise back to low freq — strong motivation to use multibit quantizers



## Multi-Stage Digital Decimation



- Sinc filter removes much of quantization noise
- Following filter(s) — anti-aliasing filter and noise



## Sinc Filter

- $\text{sinc}^{L+1}$  is a cascade of  $L+1$  averaging filters

### Averaging filter

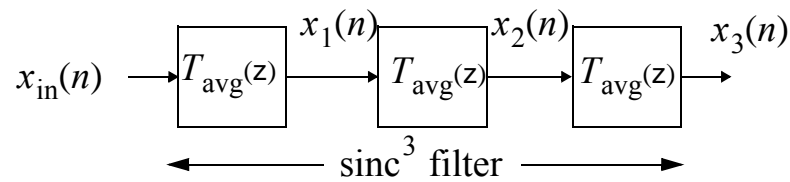
$$T_{\text{avg}}(z) = \frac{Y(z)}{U(z)} = \frac{1}{M} \sum_{i=0}^{M-1} z^{-i} \quad (21)$$

- $M$  is integer ratio of  $f_s / (8f_0)$
- It is a linear-phase filter (symmetric coefficients)
- If  $M$  is power of 2, easy division (shift left)
- Can not do all decimation filtering here since not sharp enough cutoff



## Sinc Filter

- Consider  $x_{in}(n) = \{1, 1, -1, 1, 1, -1, \dots\}$  applied to  $M = 4$  averaging filters in cascade



- $x_1(n) = \{0.5, 0.5, 0.0, 0.5, 0.5, 0.0, \dots\}$
- $x_2(n) = \{0.38, 0.38, 0.25, 0.38, 0.38, 0.25, \dots\}$
- $x_3(n) = \{0.34, 0.34, 0.31, 0.34, 0.34, 0.31, \dots\}$
- Converging to sequence of all  $1/3$  as expected



## Sinc Filter Response

- Can rewrite averaging filter in recursive form as

$$T_{avg}(z) = \frac{Y(z)}{U(z)} = \frac{1}{M} \left( \frac{1 - z^{-M}}{1 - z^{-1}} \right) \quad (22)$$

and a cascade of  $L + 1$  averaging filters results in

$$T_{sinc}(z) = \frac{1}{M^{L+1}} \left( \frac{1 - z^{-M}}{1 - z^{-1}} \right)^{L+1} \quad (23)$$

- Use  $L + 1$  cascade to roll off quantization noise faster than it rises in  $L$ 'th order modulator

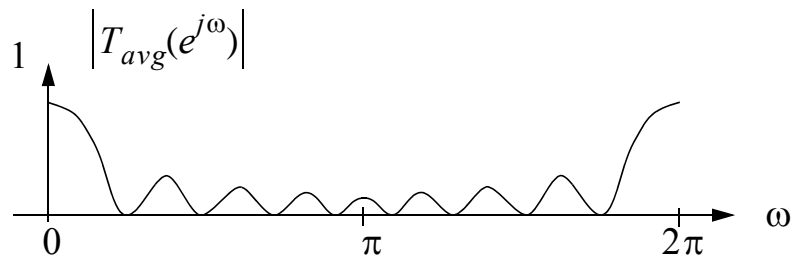


## Sinc Filter Frequency Response

- Let  $z = e^{j\omega}$

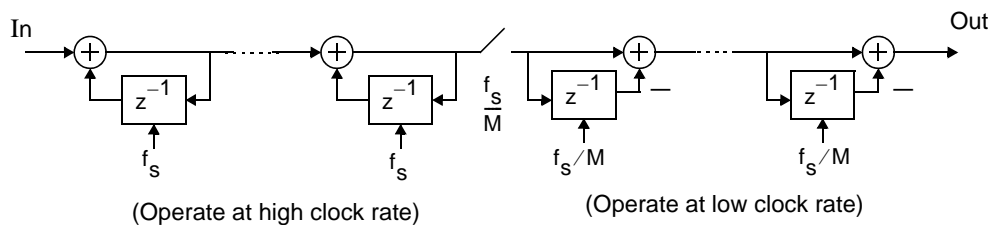
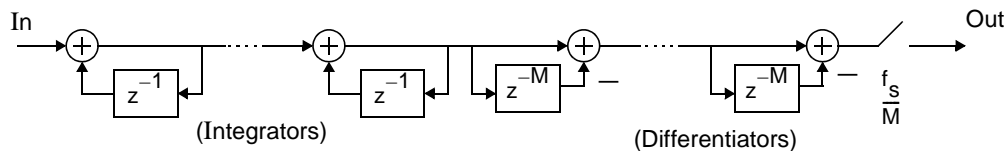
$$T_{avg}(e^{j\omega}) = \frac{\text{sinc}\left(\frac{\omega M}{2}\right)}{\text{sinc}\left(\frac{\omega}{2}\right)} \quad (24)$$

where  $\text{sinc}(x) \equiv \sin(x)/x$



## Sinc Implementation

$$T_{\text{sinc}}(z) = \left(\frac{1}{1-z^{-1}}\right)^{L+1} (1-z^M)^{L+1} \frac{1}{M^{L+1}} \quad (25)$$



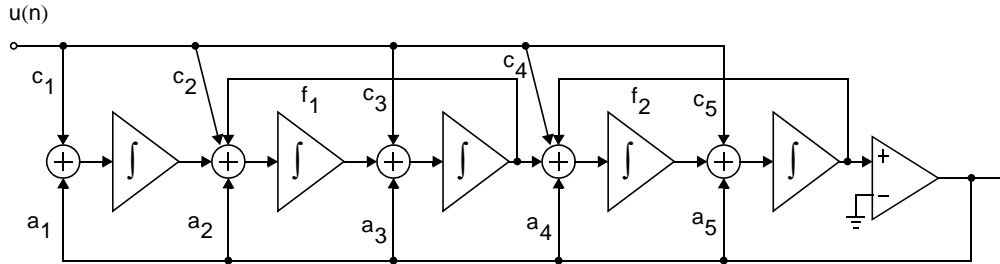
- If 2's complement arithmetic used, wrap-around okay since followed by differentiators



## Higher-Order Modulators

- An L'th order modulator improves SNR by  $6L+3$  dB/octave

### Interpolative Architecture

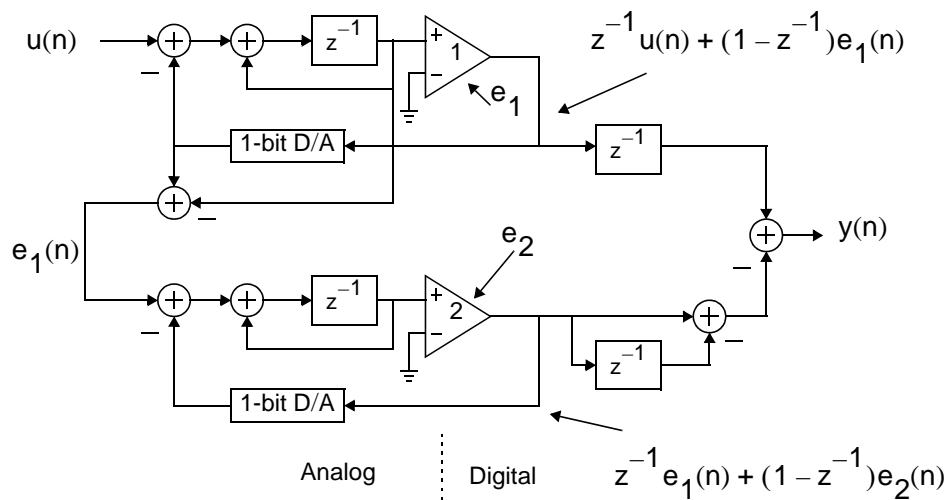


- Can spread zeros over freq of interest using resonators with  $f_1$  and  $f_2$
- Need to worry about stability (more later)



## MASH Architecture

- Multi-stage noise SHaping - MASH
- Use multiple lower order modulators and combine outputs to cancel noise of first stages



## MASH Architecture

- Output found to be:

$$Y(z) = z^{-2}U(z) - (1 - z^{-1})^2 E_2(z) \quad (26)$$

### Multibit Output

- Output is a 4-level signal though only single-bit D/A's
  - if D/A application, then linear 4-level D/A needed
  - if A/D, slightly more complex decimation

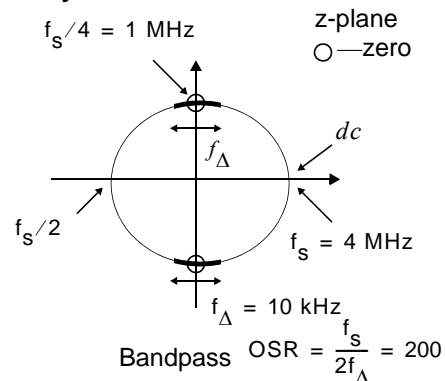
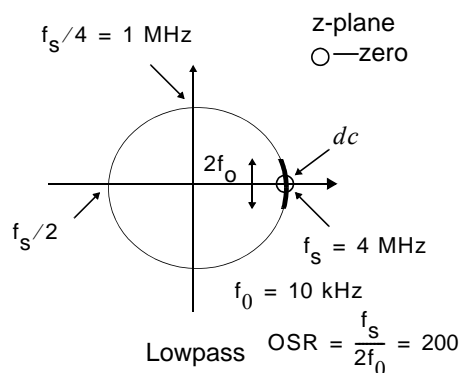
### A/D Application

- Mismatch between analog and digital can cause first-order noise,  $e_1$ , to leak through to output
- Choose first stage as higher-order (say 2'nd order)

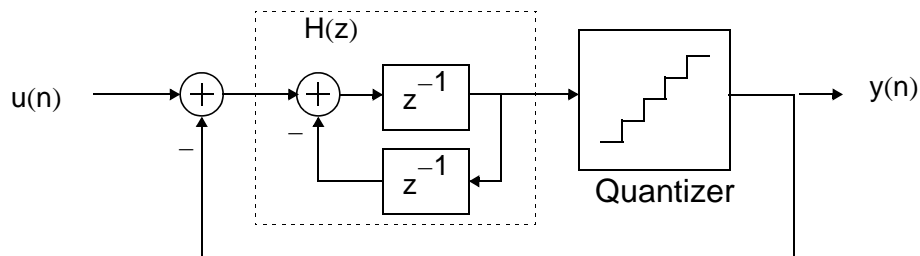


## Bandpass Oversampling Converters

- Choose  $H(z)$  to have high gain near freq  $f_c$
- NTF shapes quantization noise to be small near  $f_c$
- OSR is ratio of sampling-rate to twice bandwidth
  - not related to center frequency



## Bandpass Oversampling Converters



- Above  $H(z)$  has poles at  $\pm j$  (which are zeros of NTF)
  - $H(z)$  is a resonator with infinite gain at  $f_s/4$
  - $H(z) = z/(z^2 + 1)$
- Note one zero at  $+j$  and one zero at  $-j$ 
  - similar to lowpass first-order modulator
  - only 9 dB/octave
- For 15 dB/octave, need 4'th order BP modulator



## Modulator Stability

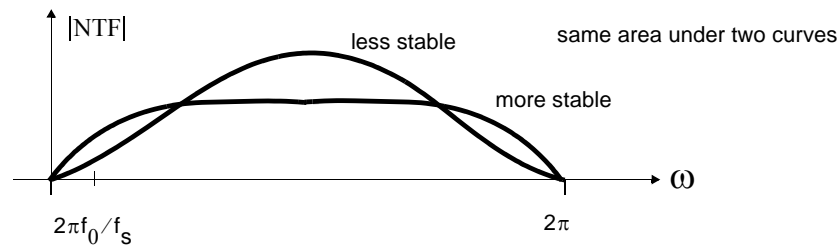
- Since feedback involved, stability is an issue
- Considered stable if quantizer input does not **overload** quantizer
- Non-trivial to analyze due to quantizer
- There are rigorous tests to guarantee stability but they are too conservative
- For a 1-bit quantizer, heuristic test is:

$$\left| N_{TF}(e^{j\omega}) \right| \leq 1.5 \quad \text{for } 0 \leq \omega \leq \pi \quad (27)$$

- Peak of NTF should be less than 1.5
- Can be made more stable by placing poles of NTF closer to its zeros
- Dynamic range suffers since less noise power pushed out-of-band



## Modulator Stability



### Stability Detection

- Might look at input to quantizer
- Might look for long strings of 1s or 0s at comp output

### When instability detected ...

- reset integrators
- Damp some integrators to force more stable

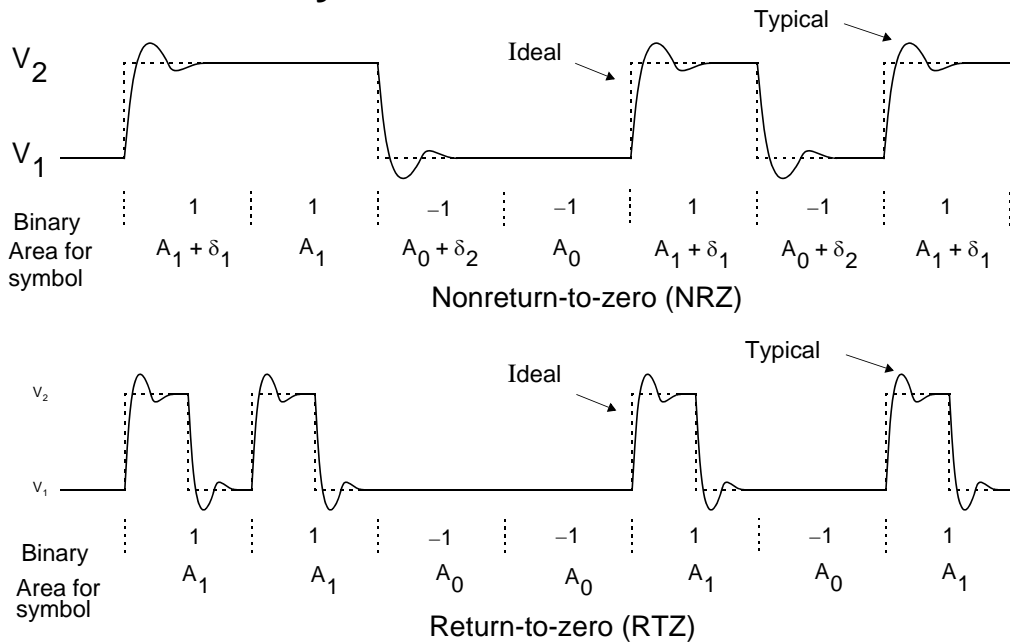


## Linearity of Two-Level Converters

- For high-linearity, levels should NOT be a function of input signal
  - power supply variation might cause symptom
- Also need to be memoryless
  - switched-capacitor circuits are inherently memoryless if enough settling-time allowed
- Above linearity issues also applicable to multi-level
- A nonreturn-to-zero is NOT memoryless
- Return-to-zero is memoryless if enough settling time
- Important for continuous-time D/A



## Linearity of Two-Level Converters



## Idle Tones

- 1/3 into 1'st order modulator results in output

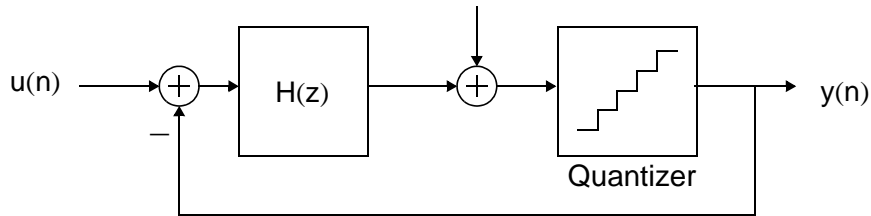
$$y(n) = \{1, 1, -1, 1, 1, -1, 1, 1, \dots\} \quad (28)$$

- Fortunately, tone is out-of-band at  $f_s/3$
- $(1/3 + 1/24) = 3/8$  into modulator has tone at  $f_s/16$
- Similar examples can cause tones in band-of-interest and are not filtered out — say  $f_s/256$
- Also true for higher-order modulators
- Human hearing can detect tones below noise floor
- Tones might not lie at single frequency but be short term periodic patterns.
  - could be a tone varying between 900 and 1100 Hz varying in a random-like pattern



## Dithering

Dither signal



- Add pseudo-random signal into modulator to break up idle tones (not just mask them)
- If added before quantizer, it is noise shaped and large dither can be added.
  - A/D: few bit D/A converter needed
  - D/A: a few bit adder needed
- Might affect modulator stability



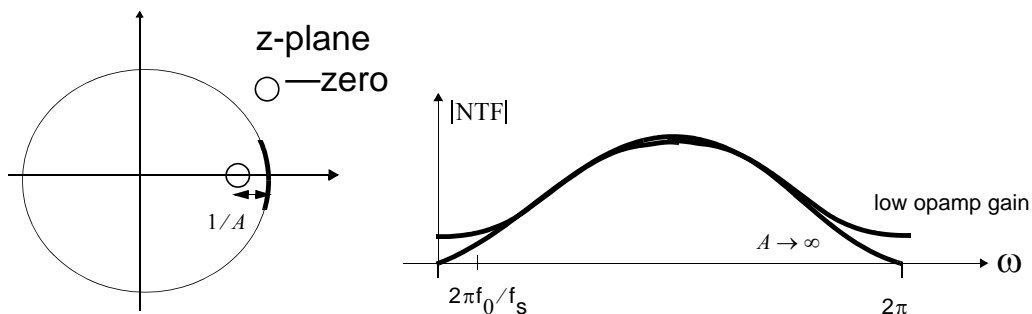
University of Toronto

slide 47 of 56

© D.A. Johns, K. Martin, 1997

## Opamp Gain

- Finite opamp gain,  $A$ , moves pole at  $z = 1$  left by  $1/A$



- Flattens out noise at low frequency
  - only 3 dB/octave for high OSR
- Typically, require

$$A > OSR / \pi \quad (29)$$



University of Toronto

slide 48 of 56

© D.A. Johns, K. Martin, 1997

## Multi-bit Oversampled Converters

- A multi-bit DAC has many advantages
  - more stable - higher peak  $|NTF|$
  - higher input range
  - less quantization noise introduced
  - less idle tones (perhaps no dithering needed)
- Need highly linear multi-bit D/A converters

### Example

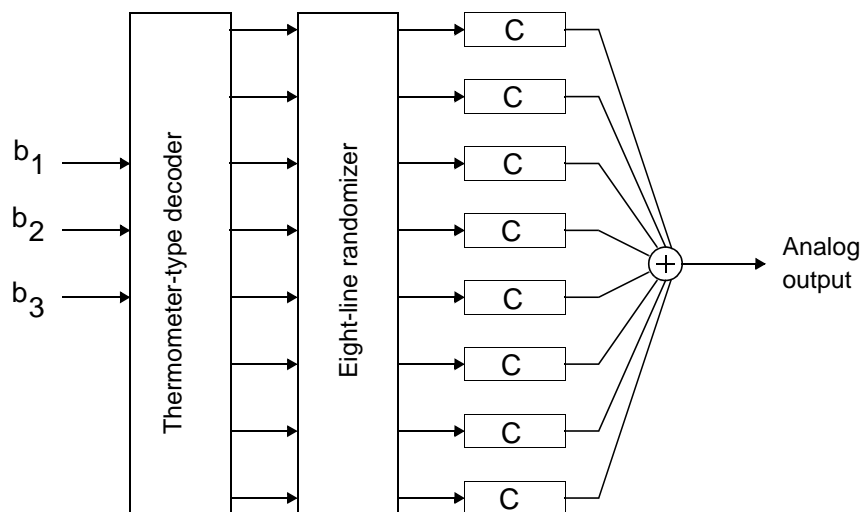
- A 4-bit DAC has 18 dB less quantization noise, up to 12 dB higher input range — perhaps 30 dB improved SNR over 1-bit

### Large Advantage in DAC Application

- Less quantization noise — easier analog lowpass filter



## Multi-bit Oversampled Converters



- Randomize thermometer code
- Can also “shape” nonlinearities



## Third-Order A/D Design Example

- All NTF zeros at  $z = 1$

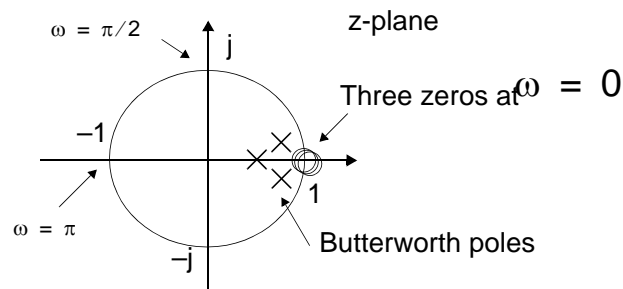
$$NTF(z) = \frac{(z-1)^3}{D(z)} \quad (30)$$

- Find  $D(z)$  such that  $|NTF(e^{j\omega})| < 1.4$
- Use Matlab to find a Butterworth highpass filter with peak gain near 1.4
- If passband edge at  $f_s/20$  then peak gain = 1.37

$$NTF(z) = \frac{(z-1)^3}{z^3 - 2.3741z^2 + 1.9294z - 0.5321} \quad (31)$$



## Third-Order A/D Design Example



- Find  $H(z)$  as

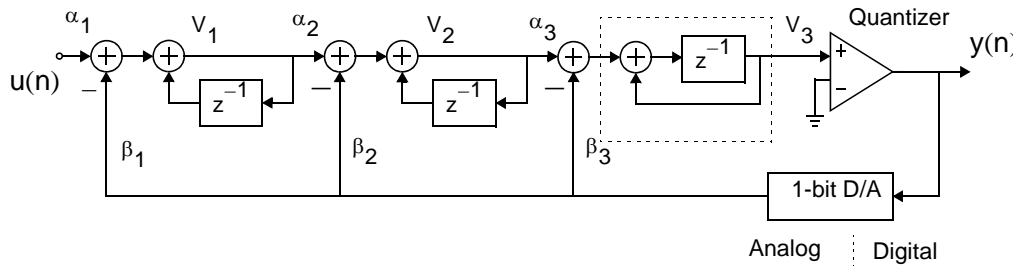
$$H(z) = \frac{1 - NTF(z)}{NTF(z)} \quad (32)$$

$$H(z) = \frac{0.6259z^2 - 1.0706z + 0.4679}{(z-1)^3} \quad (33)$$



## Third-Order A/D Design Example

- Choosing a cascade of integrator structure



- $\alpha_i$  coefficients included for dynamic-range scaling
  - initially  $\alpha_2 = \alpha_3 = 1$
  - last term,  $\alpha_1$ , initially set to  $\beta_1$  so input is stable for a reasonable input range
- Initial  $\beta_i$  found by deriving transfer function from 1-bit D/A output to  $V_3$  and equating to  $-H(z)$



## Third-Order A/D Design Example

$$H(z) = \frac{z^2(\beta_1 + \beta_2 + \beta_3) - z(\beta_2 + 2\beta_3) + \beta_3}{(z - 1)^3} \quad (34)$$

- Equating (33) and (34) results in

$$\begin{aligned} \alpha_1 &= 0.0232, & \alpha_2 &= 1.0, & \alpha_3 &= 1.0 \\ \beta_1 &= 0.0232, & \beta_2 &= 0.1348, & \beta_3 &= 0.4679 \end{aligned} \quad (35)$$



