Oversampling Converters

Motivation

• Popular approach for medium-to-low speed A/D and D/A applications requiring high resolution

Easier Analog

• reduced matching tolerances
• relaxed anti-aliasing specs
• relaxed smoothing filters

More Digital Signal Processing

• Needs to perform strict anti-aliasing or smoothing filtering
• Also removes shaped quantization noise and decimation (or interpolation)
Quantization Noise

- Above model is exact
  — approx made when assumptions made about $e(n)$
- Often assume $e(n)$ is white, uniformly distributed number between $\pm \Delta/2$
- $\Delta$ is difference between two quantization levels

Quantization Noise

- White noise assumption reasonable when:
  — fine quantization levels
  — signal crosses through many levels between samples
  — sampling rate not synchronized to signal frequency
- Sample lands somewhere in quantization interval leading to random error of $\pm \Delta/2$
Quantization Noise

- Quantization noise power shown to be $\frac{\Delta^2}{12}$ and is **independent of sampling frequency**.
- If white, then spectral density of noise, $S_n(f)$, is constant.

\[
S_n(f) = \frac{\Delta^2}{12} \frac{1}{f_s}
\]

Oversampling Advantage

- Oversampling occurs when signal of interest is bandlimited to $f_0$ but we sample higher than $2f_0$.
- Define oversampling-rate

\[
OSR = \frac{f_s}{2f_0}
\]

(1)

- After quantizing input signal, pass it through a brickwall digital filter with passband up to $f_0$.

Oversampling Advantage

• Output quantization noise after filtering is:

\[ P_e = \int_{-f_s/2}^{f_s/2} S_e^2(f) |H(f)|^2 \, df = \int_{-f_0}^{f_0} k_x^2 \, df = \frac{\Delta^2}{12} \left( \frac{1}{OSR} \right) \]  \hspace{1cm} (2)

• Doubling OSR reduces quantation noise power by 3dB (i.e. 0.5 bits/octave)

• Assuming peak input is a sinusoidal wave with a peak value of \(2^N(\Delta/2)\) leading to

\[ P_s = ((\Delta 2^N)/(2 \sqrt{2}))^2 \]

• Can also find peak SNR as:

\[ SNR_{max} = 10 \log \left( \frac{P_s}{P_e} \right) = 10 \log \left( \frac{3}{2}2^{2N} \right) + 10 \log(OSR) \]  \hspace{1cm} (3)

Example

• A dc signal with 1V is combined with a noise signal uniformly distributed between \(\pm \sqrt{3}\) giving 0 dB SNR. — \{0.94, –0.52, –0.73, 2.15, 1.91, 1.33, –0.31, 2.33\}.

• Average of 8 samples results in 0.8875

• Signal adds linearly while noise values add in a square-root fashion — noise filtered out.

Example

• 1-bit A/D gives 6dB SNR.

• To obtain 96dB SNR requires 30 octaves of oversampling ( (96-6)/3 dB/octave )

• If \(f_0 = 25 \text{ kHz} \), \(f_s = 2^{30} \times f_0 = 54,000 \text{ GHz} \)!
**Advantage of 1-bit D/A Converters**

- Oversampling improves SNR but not linearity
- To achieve 16-bit linear converter using a 12-bit converter, 12-bit converter must be linear to 16 bits — i.e. integral nonlinearity better than $1/2^4$ LSB
- A 1-bit D/A is *inherently linear* — 1-bit D/A has only 2 output points — 2 points always lie on a straight line
- Can achieve better than 20 bits linearity without trimming (will likely have gain and offset error)
- Second-order effects (such as D/A memory or signal-dependent reference voltages) will limit linearity.

**Oversampling with Noise Shaping**

- Place the quantizer in a feedback loop

\[ u(n) + \underbrace{H(z) x(n)}_{\text{Delta-Sigma Modulator}} \rightarrow y(n) \]

\[ u(n) + \underbrace{H(z) x(n)}_{\text{Linear model}} \rightarrow y(n) \]

\[ \text{Quantizer} \]

Oversampling with Noise Shaping

- Shapes quantization noise away from signal band of interest

Signal and Noise Transfer-Functions

\[ S_{TF}(z) \equiv \frac{Y(z)}{U(z)} = \frac{H(z)}{1 + H(z)} \]  \hspace{1cm} (4)

\[ N_{TF}(z) \equiv \frac{Y(z)}{E(z)} = \frac{1}{1 + H(z)} \]  \hspace{1cm} (5)

\[ Y(z) = S_{TF}(z)U(z) + N_{TF}(z)E(z) \]  \hspace{1cm} (6)

- Choose \( H(z) \) to be large over 0 to \( f_0 \)
- Resulting quantization noise near 0 where \( H(z) \) large
- Signal transfer-function near 1 where \( H(z) \) large

Oversampling with Noise Shaping

- Input signal is limited to range of quantizer output when \( H(z) \) large
- For 1-bit quantizers, input often limited to 1/4 quantizer outputs
- Out-of-band signals can be larger when \( H(z) \) small
- Stability of modulator can be an issue (particularly for higher-orders of \( H(z) \))
- Stability defined as when input to quantizer becomes so large that quantization error greater than \( \pm \Delta/2 \) — said to “overload the quantizer”
First-Order Noise Shaping

- Choose $H(z)$ to be a discrete-time integrator

$$H(z) = \frac{1}{z - 1} \quad (7)$$

- If stable, average input of integrator must be zero
- Average value of $u(n)$ must equal average of $y(n)$

Example

- The output sequence and state values when a dc input, $u(n)$, of $1/3$ is applied to a 1'st order modulator with a two-level quantizer of $\pm 1.0$. Initial state for $x(n)$ is 0.1.

<table>
<thead>
<tr>
<th>n</th>
<th>x(n)</th>
<th>x(n + 1)</th>
<th>y(n)</th>
<th>e(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>-0.5667</td>
<td>1.0</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>-0.5667</td>
<td>0.7667</td>
<td>-1.0</td>
<td>-0.4333</td>
</tr>
<tr>
<td>2</td>
<td>0.7667</td>
<td>0.1</td>
<td>1.0</td>
<td>0.2333</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>-0.5667</td>
<td>1.0</td>
<td>0.9</td>
</tr>
<tr>
<td>4</td>
<td>-0.5667</td>
<td>0.7667</td>
<td>-1.0</td>
<td>-0.4333</td>
</tr>
<tr>
<td>5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

- Average of $y(n)$ is $1/3$ as expected
- Periodic quantization noise in this case
Transfer-Functions

Signal and Noise Transfer-Functions

\[ S_{TF}(z) = \frac{Y(z)}{U(z)} = \frac{1/(z - 1)}{1 + 1/(z - 1)} = z^{-1} \] (8)

\[ N_{TF}(z) = \frac{Y(z)}{E(z)} = \frac{1}{1 + 1/(z - 1)} = (1 - z^{-1}) \] (9)

• Noise transfer-function is a discrete-time differentiator (i.e. a highpass filter)

\[ N_{TF}(f) = 1 - e^{-j2\pi f/f_s} = \frac{e^{j\pi f/f_s} - e^{-j\pi f/f_s}}{2j} \times 2j \times e^{-j\pi f/f_s} \]

\[ = \sin\left(\frac{\pi f}{f_s}\right) \times 2j \times e^{-j\pi f/f_s} \] (10)

Signal to Noise Ratio

Magnitude of noise transfer-function

\[ |N_{TF}(f)| = 2 \sin\left(\frac{\pi f}{f_s}\right) \] (11)

Quantization noise power

\[ P_e = \int_{-f_0}^{f_0} S_e^2(f)|N_{TF}(f)|^2 df = \int_{-f_0}^{f_0} \left(\frac{\Delta^2}{12}\right) \frac{f_0^2}{f_s^2} \left[2 \sin\left(\frac{\pi f}{f_s}\right)\right]^2 df \] (12)

• Assuming \( f_0 << f_s \) (i.e., \( OSR >> 1 \))

\[ P_e \simeq \left(\frac{\Delta^2}{12}\right) \left(\frac{\pi^2}{3}\right) \left(\frac{2f_0}{f_s}\right)^3 = \frac{\Delta^2 \pi^2}{36} \left(\frac{1}{OSR}\right)^3 \] (13)
Max SNR

- Assuming peak input is a sinusoidal wave with a peak value of $2^N (\Delta/2)$ leading to
  $$P_s = \left((\Delta 2^N)/(2\sqrt{2})\right)^2$$
- Can find peak SNR as:
  $$\text{SNR}_{max} = 10 \log \left(\frac{P_s}{P_e}\right)$$
  $$= 10 \log \left(\frac{3}{2} 2^N\right) + 10 \log \left[\frac{3}{\pi^2} (\text{OSR})^3\right]$$
  $$\text{SNR}_{max} = 6.02N + 1.76 - 5.17 + 30 \log (\text{OSR})$$
- Doubling OSR gives an SNR improvement 9 dB (a benefit of 1.5 bits/octave)

SC Implementation

- 1-bit D/A
- Comparator
- Latch on $\phi_2$ falling
- $V_{\text{out}}^{high}$, $V_{\text{out}}^{low}$

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Second-Order Noise Shaping

\[ S_{TF}(f) = z^{-1} \]  (16)

\[ N_{TF}(f) = (1 - z^{-1})^2 \]  (17)

\[ SNR_{max} = 6.02N + 1.76 - 12.9 + 50 \log(\text{OSR}) \]  (18)

- Doubling \( \text{OSR} \) improves SNR by 15 dB (i.e., a benefit of 2.5 bits/octave)

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Noise Transfer-Function Curves

- Out-of-band noise increases for high-order modulators
- Out-of-band noise peak controlled by poles of noise transfer-function
- Can also spread zeros over band-of-interest
Example

- 90 dB SNR improvement from A/D with $f_0 = 25$ kHz

Oversampling with no noise shaping

- From before, straight oversampling requires a sampling rate of 54,000 GHz.

First-Order Noise Shaping

- Lose 5 dB (see (15)), require 95 dB divided by 9 dB/octave, or 10.56 octaves — $f_s = 2^{10.56} \times 2f_0 \approx 75$ MHz

Second-Order Noise Shaping

- Lose 13 dB, required 103 dB divided by 15 dB/octave, $f_s = 5.8$ MHz (does not account for reduced input range needed for stability).

Quantization Noise Power of 1-bit Modulators

- If output of 1-bit mod is $\pm 1$, total power of output signal, $y(n)$, is normalized power of 1 watt.
- Signal level often limited to well below $\pm 1$ level in higher-order modulators to maintain stability
- For example, if maximum peak level is $\pm 0.25$, max signal power is 62.5 mW.
- Max signal is approx 12 dB below quantization noise (but most noise in different frequency region)
- Quantization filter must have dynamic range capable of handling full power of $y(n)$ at input.
- Easy for A/D — digital filter
- More difficult for D/A — analog filter
Zeros of NTF are poles of $H(z)$

- Write $H(z)$ as
  \[ H(z) = \frac{N(z)}{D(z)} \]  \hspace{1cm} (19)

- NTF is given by:
  \[ \text{NTF}(z) = \frac{1}{1 + H(z)} = \frac{D(z)}{D(z) + N(z)} \]  \hspace{1cm} (20)

- If poles of $H(z)$ are well-defined then so are zeros of NTF

Error-Feedback Structure

- Alternate structure to interpolative

- Signal transfer-function equals unity while noise transfer-function equals $G(z)$
- First element of $G(z)$ equals 1 for no delay free loops
- First-order system — $G(z) - 1 = -z^{-1}$
- More sensitive to coefficient mismatches
Architecture of Delta-Sigma A/D Converters

- Anti-aliasing filter
- Sample-and-hold
- ΔΣ Mod
- Digital low-pass filter
- Decimation filter
- OSR

Time

Frequency

\[ x_{in}(t) \quad x_{c}(t) \quad x_{sh}(t) \quad x_{dsm}(n) \quad x_{lp}(n) \quad x_s(n) \]

\[ x_{c}(t) \quad x_{sh}(t) \]

\[ t \]

\[ X_{c}(f) \]

\[ X_{sh}(f) \]

\[ f \]

\[ f_s \]

\[ 2f_0 \]

\[ X_{dsm}(\omega) \]

\[ X_{lp}(\omega) \]

\[ X_s(\omega) \]

\[ \omega \]

\[ 2\pi f_0 / f_s \]

\[ 2\pi \]

\[ 2 \pi f_0 / f_s \]

\[ 2\pi \]

\[ \pi \quad 2\pi \quad 4\pi \quad 6\pi \quad 8\pi \quad 10\pi \quad 12\pi \]

\[ x_{dsm}(n) = \pm 1.000000 \ldots \]

\[ x_{lp}(n) \]

\[ x_s(n) \]

\[ n \]

\[ n \]

\[ n \]

\[ n \]

\[ 2 \quad 3 \ldots \]

\[ 1 \quad 2 \quad 3 \ldots \]

\[ 1 \quad 2 \quad 3 \ldots \]

\[ \pm \]

\[ \pm \]

\[ \pm \]
Architecture of Delta-Sigma A/D Converters

- Relaxes analog anti-aliasing filter
- Strict anti-aliasing done in digital domain
- Must also remove quantization noise before downsampling (or aliasing occurs)
- Commonly done with a multi-stage system
- Linearity of D/A in modulator important — results in overall nonlinearity
- Linearity of A/D in modulator unimportant (effects reduced by high gain in feedback of modulator)

Architecture of Delta-Sigma D/A Converters

\[ O.S.R = \frac{f_s}{2f_0} \]

\[ X_s(n), X_{s2}(n), X_{lp}(n), X_{dsm}(n), X_{da}(t), X_c(t) \]

Time

\[ \omega \]

Frequency
Architecture of Delta-Sigma D/A Converters

- Relaxes analog smoothing filter (many multibit D/A converters are oversampled without noise shaping)
- Smoothing filter of first few images done in digital (then often below quantization noise)
- Order of lowpass filter should be at least one order higher than that of modulator
- Results in noise dropping off (rather than flat)
- Analog filter must attenuate quantization noise and should not modulate noise back to low freq — strong motivation to use multibit quantizers
Multi-Stage Digital Decimation

- Sinc filter removes much of quantization noise
- Following filter(s) — anti-aliasing filter and noise

Sinc Filter

- sinc\textsuperscript{L+1} is a cascade of \(L+1\) averaging filters

\textbf{Averaging filter}

\[ T_{\text{avg}}(z) = \frac{Y(z)}{U(z)} = \frac{1}{M} \sum_{i=0}^{M-1} z^{-i} \]  

\( M \) is integer ratio of \( f_s/(8f_0) \)
- It is a linear-phase filter (symmetric coefficients)
- If \( M \) is power of 2, easy division (shift left)
- Can not do all decimation filtering here since not sharp enough cutoff
### Sinc Filter

- Consider $x_{in}(n) = \{1, 1, -1, 1, 1, -1, \ldots\}$ applied to $M = 4$ averaging filters in cascade

\[
x_{in}(n) \xrightarrow{T_{avg}(z)} x_{1}(n) \xrightarrow{T_{avg}(z)} x_{2}(n) \xrightarrow{T_{avg}(z)} x_{3}(n)
\]

- $x_{1}(n) = \{0.5, 0.5, 0.0, 0.5, 0.5, 0.0, \ldots\}$
- $x_{2}(n) = \{0.38, 0.38, 0.25, 0.38, 0.38, 0.25, \ldots\}$
- $x_{3}(n) = \{0.34, 0.34, 0.31, 0.34, 0.34, 0.31, \ldots\}$
- Converging to sequence of all $1/3$ as expected

\[
sinc^3 \text{ filter}
\]

### Sinc Filter Response

- Can rewrite averaging filter in recursive form as

\[
T_{avg}(z) = \frac{Y(z)}{U(z)} = \frac{1}{M} \left( \frac{1 - z^{-M}}{1 - z^{-1}} \right)
\]  \hspace{1cm} (22)

and a cascade of $L + 1$ averaging filters results in

\[
T_{sinc}(z) = \frac{1}{M^{L+1}} \left( \frac{1 - z^{-M}}{1 - z^{-1}} \right)^{L+1}
\]  \hspace{1cm} (23)

- Use $L + 1$ cascade to roll off quantization noise faster than it rises in $L$‘th order modulator
Sinc Filter Frequency Response

- Let \( z = e^{j\omega} \)

\[
T_{avg}(e^{j\omega}) = \frac{sinc\left(\frac{\omega M}{2}\right)}{sinc\left(\frac{\omega}{2}\right)}
\]  

(24)

where \( sinc(x) \equiv \frac{\sin(x)}{x} \)

\[
|T_{avg}(e^{j\omega})|
\]

\[\begin{align*}
0 & \quad \pi \\
& \quad 2\pi
\end{align*}\]

Sinc Implementation

\[
T_{sinc}(z) = \left( \frac{1}{1 - z^{-1}} \right)^{L+1} (1 - z^{-M})^{L+1} \frac{1}{M^{L+1}}
\]  

(25)

- If 2's complement arithmetic used, wrap-around okay since followed by differentiators

\[
\begin{array}{c}
\text{(Integrators)} \\
\text{(Differentiators)}
\end{array}
\]

(Operate at low clock rate)

(Operate at high clock rate)
Higher-Order Modulators

- An L’th order modulator improves SNR by 6L+3 dB/octave

Interpolative Architecture

- Can spread zeros over freq of interest using resonators with \( f_1 \) and \( f_2 \)
- Need to worry about stability (more later)

MASH Architecture

- Multi-stAge noise SHaping - MASH
- Use multiple lower order modulators and combine outputs to cancel noise of first stages
**MASH Architecture**

- Output found to be:

\[ Y(z) = z^{-2} U(z) - (1 - z^{-1})^2 E_2(z) \]  

(26)

**Multibit Output**

- Output is a 4-level signal though only single-bit D/A’s
  - if D/A application, then linear 4-level D/A needed
  - if A/D, slightly more complex decimation

**A/D Application**

- Mismatch between analog and digital can cause first-order noise, \( e_1 \), to leak through to output
- Choose first stage as higher-order (say 2’nd order)

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**Bandpass Oversampling Converters**

- Choose \( H(z) \) to have high gain near freq \( f_c \)
- NTF shapes quantization noise to be small near \( f_c \)
- OSR is ratio of sampling-rate to twice bandwidth — not related to center frequency

\[ \frac{f_s}{2f_0} = 200 \text{ Lowpass} \]

\[ \frac{f_s}{2f_\Delta} = 200 \text{ Bandpass} \]
Bandpass Oversampling Converters

- Above $H(z)$ has poles at $\pm j$ (which are zeros of NTF)
  - $H(z)$ is a resonator with infinite gain at $f_s/4$
  - $H(z) = z / (z^2 + 1)$
- Note one zero at $+j$ and one zero at $-j$
  - similar to lowpass first-order modulator
  - only 9 dB/octave
- For 15 dB/octave, need 4'th order BP modulator

Modulator Stability

- Since feedback involved, stability is an issue
- Considered stable if quantizer input does not overload quantizer
- Non-trivial to analyze due to quantizer
- There are rigorous tests to guarantee stability but they are too conservative
- For a 1-bit quantizer, heuristic test is:
  \[ |N_{TF}(e^{j\omega})| \leq 1.5 \quad \text{for } 0 \leq \omega \leq \pi \]  
  (27)
- Peak of NTF should be less than 1.5
- Can be made more stable by placing poles of NTF closer to its zeros
- Dynamic range suffers since less noise power pushed out-of-band
Modulator Stability

Stability Detection
- Might look at input to quantizer
- Might look for long strings of 1s or 0s at comp output

When instability detected ...
- reset integrators
- Damp some integrators to force more stable

Linearity of Two-Level Converters
- For high-linearity, levels should NOT be a function of input signal
  — power supply variation might cause symptom
- Also need to be memoryless
  — switched-capacitor circuits are inherently memoryless if enough settling-time allowed
- Above linearity issues also applicable to multi-level
- A nonreturn-to-zero is NOT memoryless
- Return-to-zero is memoryless if enough settling time
- Important for continuous-time D/A
**Linearity of Two-Level Converters**

![Diagram of two-level converters](image)

- **Nonreturn-to-zero (NRZ)**
  - Binary Area for symbol: \( A_1 + \delta_1 \), \( A_1 \), \( A_0 + \delta_2 \), \( A_0 \), \( A_1 + \delta_1 \), \( A_1 + \delta_1 \)

- **Return-to-zero (RTZ)**
  - Binary Area for symbol: \( A_1 \), \( A_1 \), \( A_0 \), \( A_0 \), \( A_1 \), \( A_0 \), \( A_1 \)

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**Idle Tones**

- 1/3 into 1’st order modulator results in output
  \[ y(n) = \{ 1, 1, -1, 1, 1, -1, 1, 1, \ldots \} \] (28)

- Fortunately, tone is out-of-band at \( f_s / 3 \)

- \((1/3 + 1/24) = 3/8\) into modulator has tone at \( f_s / 16 \)

- Similar examples can cause tones in band-of-interest and are not filtered out — say \( f_s / 256 \)

- Also true for higher-order modulators

- Human hearing can detect tones below noise floor

- Tones might not lie at single frequency but be short term periodic patterns.
  — could be a tone varying between 900 and 1100 Hz varying in a random-like pattern
**Dithering**

Dither signal

- Add pseudo-random signal into modulator to break up idle tones (not just mask them)
- If added before quantizer, it is noise shaped and large dither can be added.
  - A/D: few bit D/A converter needed
  - D/A: a few bit adder needed
- Might affect modulator stability

**Opamp Gain**

- Finite opamp gain, $A$, moves pole at $z = 1$ left by $1/A$
- Flattens out noise at low frequency
  - only 3 dB/octave for high OSR
- Typically, require

$$A > \frac{OSR}{\pi}$$  \hspace{1cm} (29)
Multi-bit Oversampled Converters

- A multi-bit DAC has many advantages
  — more stable - higher peak |NTF|
  — higher input range
  — less quantization noise introduced
  — less idle tones (perhaps no dithering needed)
- Need highly linear multi-bit D/A converters

Example

- A 4-bit DAC has 18 dB less quantization noise, up to 12 dB higher input range — perhaps 30 dB improved SNR over 1-bit

Large Advantage in DAC Application

- Less quantization noise — easier analog lowpass filter

Multi-bit Oversampled Converters

- Randomize thermometer code
- Can also “shape” nonlinearities
Third-Order A/D Design Example

- All NTF zeros at $z = 1$
  \[ NTF(z) = \frac{(z - 1)^3}{D(z)} \quad (30) \]

- Find $D(z)$ such that $|NTF(e^{j\omega})| < 1.4$
- Use Matlab to find a Butterworth highpass filter with peak gain near 1.4
- If passband edge at $f_s/20$ then peak gain = 1.37
  \[ NTF(z) = \frac{(z - 1)^3}{z^3 - 2.3741z^2 + 1.9294z - 0.5321} \quad (31) \]

- Find $H(z)$ as
  \[ H(z) = \frac{1 - NTF(z)}{NTF(z)} \quad (32) \]
  \[ H(z) = \frac{0.6259z^2 - 1.0706z + 0.4679}{(z - 1)^3} \quad (33) \]
Third-Order A/D Design Example

- Choosing a cascade of integrator structure

\[ H(z) = \frac{z^2(\beta_1 + \beta_2 + \beta_3) - z(2\beta_2 + 2\beta_3) + \beta_3}{(z - 1)^3} \]  

(34)

- Equating (33) and (34) results in

\[ \begin{align*}
\alpha_1 &= 0.0232, \quad \alpha_2 = 1.0, \quad \alpha_3 = 1.0 \\
\beta_1 &= 0.0232, \quad \beta_2 = 0.1348, \quad \beta_3 = 0.4679
\end{align*} \]  

(35)

- \(\alpha_i\) coefficients included for dynamic-range scaling
  - initially \(\alpha_2 = \alpha_3 = 1\)
  - last term, \(\alpha_1\), initially set to \(\beta_1\) so input is stable for a reasonable input range

- Initial \(\beta_i\) found by deriving transfer function from 1-bit D/A output to \(V_3\) and equating to \(-H(z)\)
Third-Order A/D Design Example

Dynamic Range Scaling

- Apply sinusoidal input signal with peak value of 0.7 and frequency $\pi/256$ rad/sample
- Simulation shows max values at nodes $V_1$, $V_2$, $V_3$ of 0.1256, 0.5108, and 1.004
- Can scale node $V_1$ by $k_1$ by multiplying $\alpha_1$ and $\beta_1$ by $k_1$ and dividing $\alpha_2$ by $k_1$
- Can scale node $V_2$ by $k_2$ by multiplying $\alpha_2/k_1$ and $\beta_2$ by $k_2$ and dividing $\alpha_3$ by $k_2$

$$\alpha_1' = 0.1847, \quad \alpha_2' = 0.2459, \quad \alpha_3' = 0.5108$$
$$\beta_1' = 0.1847, \quad \beta_2' = 0.2639, \quad \beta_3' = 0.4679$$

(36)