CHEBYSHEV LOWPASS

Let \( F(w^2) = e^{2CN(w)} \)

where \( e \) is small number

\[
CN(w) = \cos (N \cos^{-1}(w)) \quad |w| \leq 1
\]
\[
CN(w) = \cosh (N \cosh^{-1}(w)) \quad |w| \geq 1
\]

\( CN(w) \) is Chebyshev polynomial of \( N \)th order and is a polynomial in \( w \) of \( N \)th degree.

To see that let \( \cos \theta = w \)

\[
\theta = \cos^{-1} w
\]

\[
CN+1(w) = \cos ((N+1)\theta) = \cos (N\theta + \theta)
\]

\[\text{1.} \quad CN+1(w) = \cos (N\theta)\cos(\theta) - \sin(N\theta)\sin(\theta)\]

\[\text{2.} \quad CN-1(w) = \cos (N\theta)\cos(\theta) + \sin(N\theta)\sin(\theta)\]

\[\text{1+2.} \quad CN+1(w) + CN-1(w) = 2 \cos (N\theta) \cos(\theta)\]

\[\theta \cos N\theta = \cos (N \cos^{-1}(w)) = CN(w)\]

\[\theta \cos \theta = w\]

\[\theta \cos (N \theta) + CN-1(w) = 2w \cos (N \cos^{-1}(w)) = \]

\[\theta \cos (N \theta) + CN-1(w) = 2w \cos (CN(w))\]
\[ C_{n+1}(w) = 2w C_n(w) - C_{n-1}(w) \]

(Similar for \( \text{cosh}(N \text{cosh}(w)) \))

Recursive formula to find \( C_n(w) \)

\[ C_0(w) = \cos(0) = 1 \]
\[ C_1(w) = \cos(\cos^{-1}(w)) = w \]
\[ C_2(w) = 2w C_1(w) - C_0(w) \]
\[ = 2w^2 - 1 \]
\[ C_3(w) = 4w^3 - 3w \]
\[ C_4(w) = 8w^4 - 8w^2 + 1 \]
\[ C_5(w) = 16w^5 - 20w^3 + 5w \]
\[ C_6(w) = 32w^6 - 48w^4 + 18w^2 - 1 \]
For \( |w| \leq 1 \) \( |C_n(w)| \leq 1 \)

For \( w > 1 \) \( C_n(w) > 1 \)

+ Monotonically rising

\( \text{For } w > 1 \) \( C_n(w) \approx 2^{n-1} w^n \)

\[
|H(iw)|^2 = \frac{A_0^2}{1 + \varepsilon^2 \cos^2[N \cos^{-1}(\frac{w}{w_p})]} \quad w \leq w_p
\]

\[
|H(iw)|^2 = \frac{A_0^2}{1 + \varepsilon^2 \cos^2[N \cos^{-1}(\frac{w}{w_p})]} \quad w \geq w_p
\]
\[ A_{\text{max}} = 10 \log (1 + e^2) \]
\[ e = \sqrt{10 \frac{A_{\text{max}}}{10} - 1} \]

To find \( e \)

\[ A_{\text{min}} \leq 10 \log \left[ 1 + e^2 \coth^2 \left( N \coth^{-1} \left( \frac{w_5}{w_p} \right) \right) \right] \]

To find \( N \)

Poles on an ellipse located at

\[ P_k = -w_p \sin \left( \frac{2k-1}{N} \frac{\pi}{2} \right) \sinh \left( \frac{1}{N} \sinh^{-1} \frac{1}{e} \right) \]
\[ + jw_p \cos \left( \frac{2k-1}{N} \frac{\pi}{2} \right) \cosh \left( \frac{1}{N} \sinh^{-1} \frac{1}{e} \right) \]

\( k = 1, 2, \ldots, N \)

\[ H(s) = \frac{k w_p^N}{(s-p_1)(s-p_2) \ldots (s-p_N)} \]

\( k \) is the dc gain
Comparison of Butterworth and Chebyshev

**Butterworth**

\[ F(w^2) = w^{2n} \quad \text{Let } w_0 = 1 \]

\[ |H(iw)|^2 = \frac{1}{1 + \epsilon^2 w^{2n}} \]

For \( w >> 1 \)

\[ |H(iw)|^2 \approx \frac{1}{\epsilon^2 w^{2n}} \]

**Chebyshev**

\[ F(w^2) = \epsilon^2 c_n^2(w) \quad \text{Let } w_0 = 1 \]

\[ |H(iw)|^2 = \frac{1}{1 + \epsilon^2 c_n^2(w)} \]

For \( w >> 1 \)

\[ |H(iw)| \approx \frac{1}{\epsilon^2 \frac{2^{(n-1)}}{w^{2n}}} \quad \text{From (3)} \]

Which is \( \frac{1}{2^{(n-1)}} \) times smaller.

\[ \text{or} \quad 10 \log_2 2^{(n-1)} = 20(n-1) \log_2 \]

\[ = 6(n-1) \text{ dB} \quad \text{more attenuation} \]
So if \( N=7 \) a Chebyshev

would have \( \epsilon = 6(7-1) = 36 \) dB

more attenuation when \( \omega >> \omega_p \)

than a Butterworth

Example

\[ A_{\text{max}} = 1 \text{ dB} \]
\[ A_{\text{min}} = 25 \text{ dB} \]
\[ \omega_p = 1 \text{ rad/s} \]
\[ \omega_s = 1.5 \text{ rad/s} \]

Find \( N + \frac{\epsilon}{4\epsilon^2} \) for Chebyshev

\[ \epsilon = \sqrt{10^{\frac{A_{\text{max}}}{10}} - 1} = 0.5088 \]

\[ 25 = 10 \log \left[ 1 + \epsilon^2 \cosh^2 \left[ N \cosh^{-1} \left( \frac{\omega_s}{\omega_p} \right) \right] \right] \]

\[ \frac{2.5}{(0.5088)^2} = \cosh^2 \left[ N \cosh^{-1} (1.5) \right] \]

\[ 34.89 = \cosh \left[ N \cosh^{-1} (1.5) \right] \]
\[ N = \frac{\text{Cosh}^{-1}[3.489]}{\text{Cosh}^{-1}[1.5]} \]

= 4.41

Choose \( N = 5 \)

\[ |H(iw)|^2 = \frac{1}{1 + (0.5088)^2 (16w^5 - 20w^3 + 5w)^2} \]

Can also build

- Elliptic filters
- Bessel-Thomson

![Graph showing elliptic filter characteristics](image)

Elliptic Ripples in Passband

(Not all-pole)
Some Filter plots - magnitude

\[ |H(j\omega)| \]

Chebyshev

Bessel-Thomson

Elliptic

Butterworth

\[ \omega \]

Some Filter plots - group delay

\[ T_d(\omega) \]

Elliptic

Chebyshev

Bessel-Thomson

Butterworth

\[ \omega \]
FIRST AND SECOND ORDER FILTERS

WHY SO IMPORTANT?

A N’TH ORDER H(S) CAN BE FACTORED INTO 1’ST ORDER TRANSFER-FUNCTIONS

\[ H(s) = H_1(s)H_2(s) \ldots H_N(s) \]

WHERE \( H_i(s) \) ARE 1’ST ORDER

\[ \text{VIN} \rightarrow H(s) \rightarrow \text{VOUT} \]

\[ \text{VIN} \rightarrow H_1(s) \rightarrow H_2(s) \rightarrow \ldots \rightarrow H_N(s) \rightarrow \text{VOUT} \]

BUT IF H(S) HAS COMPLEX ZEROS OR POLES

THEN THOSE H_i(S) HAVE COMPLEX COEFFICIENTS \( \Rightarrow \) NOT EASILY REALIZED
However, if we group complex-conjugate pairs of poles and zeros into 2nd order sections then all coefficients become real.

Specifically, if

\[ H(s) = \frac{[s-(x_1+jy_1)]\cdots[x_1+jy_1]}{[s-(x_2+jy_2)]\cdots[x_2+jy_2]} \]

\[ H \{ s \} \text{ REAL} \]

\[ T_1(s), T_2(s) \]

\[ H_{BQ} \{ s \} \]

\[ H(s) = \frac{[s^2-2xs-(x^2+y^2)]}{[s^2-2xs-(x^2+y^2)]} \cdots \]

\[ H_{BQ} \{ s \} \beta \text{ Biquad} \]

Can always write \( H(s) \) in factored form containing biquads (with only real coeff) and possibly one bilinear 1st order with real coeff.

\[ H(s) = \frac{a_1s^2+a_0}{s^2+b_0} \]

\[ \text{DC} \quad H(0) = \frac{a_0}{b_0} \quad \text{at} \quad H(\infty) = a_1 \]

\[ \text{transmission zero at} \quad s = -\frac{a_0}{a_1} \]

\[ \text{poles at} \quad s = -\frac{a_1s^2+a_0}{s^2+b_0} \quad \text{frequency} \quad w_0 \]

\[ \text{marker pole at} \quad s = -w_0 \quad w_0 > 0 \quad \text{for stability} \]

\[ \frac{a_1s^2+a_0}{s^2+b_0} \]

\[ w_0 \rightarrow \text{pole} \]
a) Low Pass \[ H(s) = \frac{a_0}{s + w_0} \]

DBE Plot

-20 dB/dec

- \infty \rightarrow \frac{s}{w_0}

\[ \frac{v_i}{v_o} = R \]

\[ C = \frac{1}{w_0} \]

DC Gain = 1

Active Realization

So can cascade circuits with little interaction

b) High Pass \[ H(s) = \frac{a_1 s}{s + w_0} \]

DB Plot

- \infty \rightarrow \frac{s}{w_0}

\[ \frac{v_i}{v_o} = \frac{C}{R} \]

\[ C = \frac{1}{w_0} \]

NF Gain = 1

Active Realization

See text for general & all-pass cases.
(8) Biquadratic \[ H(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + 6s + 60} = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + (\omega_0/s) s + \omega_0^2} \]

**Poles**

\[ \omega_0 = \sqrt{b_0} \]
**Pole Frequency**

\[ q = \frac{\sqrt{b_0}}{b_1} \]
**Pole Quality Factor (or Q factor)**

Writing \[ s^2 + \frac{\omega_0}{q} s + \omega_0^2 = (s - e_1)(s - e_2) \] (i.e., solving for roots)

\[ e_1, e_2 = -\frac{\omega_0}{2q} \pm j\omega_0 \sqrt{1 - \frac{1}{4q^2}} \]

- If \( \frac{1}{4q^2} < 1 \) OR \( q > 0.5 \), \( e_1 \) and \( e_2 \) are complex-conjugate
- \( |e_1| = |e_2| = \omega_0 \) and \( \text{REAL}[e_1] = \text{REAL}[e_2] = -\frac{\omega_0}{2q} \)

**Graphically**

For \( q > 0.5 \)

As \( q \to \infty \), \( e_1 \) and \( e_2 \) approach \( j \omega_0 \) axis

For \( q < 0.5 \)

\( e_1 \) and \( e_2 \) are on the real axis

\( q > 0 \) for stability
Transmission Zeros

\[ \frac{a_2 s^2 + a_1 s + a_0}{b_p \Rightarrow a_1 s \quad \text{Notch} \Rightarrow a_2 s^2 + a_0} \]

\[ L \rho \Rightarrow a_0 \quad H \rho \Rightarrow a_2 s^2 \]

\text{a) Low Pass}\n\[ H(s) = \frac{a_0}{s^2 + \frac{a_1}{a_0} s + w_0^2} \]
\[ \text{DC Gain} \Rightarrow T(0) = \frac{a_0}{w_0^2} \]

\[ \text{Peak only if} \quad Q > \frac{1}{\sqrt{a_1}} \]
\[ Q < \frac{1}{\sqrt{a_1}} \]

\[ \text{Max} = W_0 \sqrt{1 - \frac{1}{a_1}} \]

\text{b) High Pass}\n\[ H(s) = \frac{a_2 s^2}{s^2 + \frac{a_1}{a_2} s + w_0^2} \]
\[ \text{HF Gain} \Rightarrow H(w) = a_2 \]

\[ \text{Max} = W_0 \sqrt{1 - \frac{1}{a_1}} \]

\text{c) Band Pass}\n\[ H(s) = \frac{a_1 s}{s^2 + \frac{a_1}{a_2} s + w_0^2} \]

\[ \text{Width of 3 dB Bandwidth} \quad \text{BW} = \frac{w_0}{Q} \]

\[ 3 \text{dB Bandwidth} = W_2 - W_1 \quad \text{where} \quad H(W_2) = |H(W_1)| = \frac{1}{\sqrt{2}} \text{ peak value} \]
d) **Low Pass Notch (LPN)**

\[ H(s) = a_2 \frac{s^2 + w_n^2}{s^2 + \frac{w_0}{a}s + w_n^2} \quad w_n \geq w_0 \]

\[ H(0) = \frac{a_2w_n^2}{w_0^2} \]

\[ H(\infty) = a_2 \]

\[ \text{Note: } \Phi = 0 \]

\[ w_0 \quad w_n \]

\[ w \]

---

e) **High Pass Notch (HPN)**

\[ H(s) = a_2 \frac{s^2 + w_n^2}{s^2 + \frac{w_0}{a}s + w_n^2} \quad w_n \leq w_0 \]

\[ H(0) = \frac{a_2w_n^2}{w_0^2} \]

\[ H(\infty) = a_2 \]

\[ w_0 \quad w_n \]

\[ w \]

---

f) **All-pass**

\[ H(s) = a_2 \frac{s^2 - \frac{w_0}{a}s + w_0^2}{s^2 + \frac{w_0}{a}s + w_0^2} \]

\[ \Phi(\omega) \]

\[ 180^\circ \quad -180^\circ \]

\[ \omega \]

\[ T_d(\omega) \]
Note for phase: the effect of a zero in the RHP is the same as a mirror image pole in LHP.

\[ \text{SAME PHASE AS} \]

2nd order LCR resonator

The poles of a biquad can be realized with the following LCR circuit.

\[ \text{SHALL SEE THAT} \quad w_0 = \frac{1}{\sqrt{LC}} \]

\[ Q = \frac{w_0}{\sqrt{LCR}} \]

If the input is applied or output taken such that it does not affect the natural resonance, then the poles can be found as the denominator of the transfer function polynomial.
WHY CARE ABOUT LCR REALIZATIONS?

IT TURNS OUT THAT MANY FILTER REALIZATIONS SUCH AS ACTIVE-RC OR DIGITAL FILTERS CAN EMULATE THE OPERATION OF LCR FILTERS AS THEY ARE NEAR OPTIMAL.

EXAMPLES

\[ \frac{V_{\text{out}}}{I_{\text{in}}} = Z = \left( \frac{1}{R} + \frac{1}{sL} \right) \]

\[ = \frac{s/C}{s^2 + \frac{1}{CR} + \frac{1}{LC}} \]

\[ = \frac{s/C}{s^2 + s \frac{1}{CR} + \frac{1}{LC}} \]

\[ = \frac{s/C}{s^2 + \frac{1}{CR} + \frac{1}{LC}} \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ Q = \frac{\omega_0}{\sqrt{CR}} \]

\[ Q = \omega_0 CR \]

Also bandpass

Center freq. gain of 1

Bandpass since \( \omega_0 @ 0 \)

Center freq. gain of \( R \)
FOR TRANSMISSION ZEROS

CONSIDER

\[ H(s) = \frac{V_{out}}{V_{in}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \]

TRANSMISSION ZEROS OCCUR WHERE \( Z_1(s) = 0 \) OR \( Z_2(s) = \infty \)

\(+\) WHERE \( Z_2(s) = 0 \) OR \( Z_1(s) = \infty \)

LOWPASS

\[ V_{out} = \frac{V_{in}}{s^2 + \frac{1}{LC} + \frac{1}{CR}} \]

HIGH PASS

\[ V_{out} = \frac{5^2}{s^2 + \frac{1}{LC} + \frac{1}{CR}} \]

BAND PASS

\[ V_{out} = \frac{5}{s^2 + \frac{1}{CR} + \frac{1}{LC}} \]

NOTE 2: \( Z_1 \neq 0 \) OR \( \infty \) FOR ANY \( \omega \)

BUT \( Z_2 = 0 \) AT \( \omega = 0 \) OR \( \infty \)
GENERAL NOTCH ZER0 PLACE AT \( \pm jw_0 \)

\[
\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + \frac{s^2 + w_0^2}{\omega_c^2}}
\]

\( z(s) \to \infty \) @ \( w = \pm jw_0 \) ie select \( L_1C_1 = \frac{1}{w_0^2} \)

TO NOT AFFECT POLES: SHORT \( V_{\text{in}} \) TO RECOGNIZE

\[
\begin{align*}
L_1/L_2 &= L \\
1/C_1 + C_2 &= C \\
\omega_0 &= \frac{1}{\sqrt{LC}} \\
\alpha &= \omega_0 CR
\end{align*}
\]

NOTE THAT

\[
\begin{align*}
H(0) &= \frac{L_2}{L_1 + L_2} \\
H(\infty) &= \frac{C_1}{C_1 + C_2}
\end{align*}
\]

FOR LOW PASS NOTCH WITH \( H(0) = 1 \) LET \( L_2 = \infty \)

\[
\begin{align*}
\frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{s^2 + w_0^2}{\omega_c^2} \\
\frac{1}{L(C_1 + C_2)} &= \omega_0 (C_1 + C_2)R \\
\omega_0 &= \frac{1}{\sqrt{C_1 + C_2}} \\
\omega_0^2 &= \frac{1}{L(C_1 + C_2)} \\
\omega_n^2 &= \frac{1}{L_1C} \\
\alpha_2 &= \frac{C_1}{C_1 + C_2} = H(\infty)
\end{align*}
\]

HIGH PASS NOTCH \( H(\infty) = 1 \) SO LET \( \alpha = 0 \)

\[
\begin{align*}
\omega_n^2 &= \frac{1}{L_1C} \\
\omega_0^2 &= \frac{1}{(L_1L_2)C} \\
\alpha &= \omega_0 CR
\end{align*}
\]
ALL-PASS

\[ H(s) = \frac{s^2 - s \frac{\omega_0}{Q} + \omega_0^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} = 1 - \frac{s}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

**Bandpass with center freq gain of 2.**

ALREADY HAVE BANDPASS WITH CENTER FREQUENCY GAIN OF 2.

SO DESIGN ALL-PASS WITH MAGNITUDE GAIN OF 0.5.

\[ H(s) = 0.5 - \frac{s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \]

ONE REALIZATION

\[ \omega_0 = \frac{1}{LC} \quad Q = \omega_0 CR \]

\[ |H(i\omega)| = 0.5 \]
**General Case**

Include \( R_L + R_C \)

\[
\frac{1}{sL + R_L} \parallel \frac{1}{sC + R_C} \parallel R
\]

\[
= \left( \frac{1}{sL + R_L} + \frac{sC}{1 + sCR_C} + \frac{1}{R} \right)^{-1}
\]

\[
= R(1 + sCR_C) + sCR(sL + R_L) + (sL + R_L)(1 + sCR_C)
\]

\[
= R + sCR + s^2LCR + sCR + R_L + s^2LCR + sCR_C
\]

\[
= s^2(LCR + LCR_C) + s(CR + CR + L + CR_C) + R + R_L
\]

\[
\omega_0 = \frac{R + R_L}{LCR + LCR_C} = \frac{R + R_L}{LC(R + R_C)}
\]

\[
Q = \frac{\omega_0 LC(R + R_C)}{CR(R + R_C) + CR_CRC + L}
\]
Lowpass Current Input

\[ H(s) = \frac{V_{out}}{V_{in}} \]

\[ H(j\omega) = \frac{R_1 R_L}{R + R_L} \]

Lowpass with Source Resistance, \( R_S \)

\[ H(s) = \frac{V_{out}}{V_{in}} \]

\[ H(j\omega) = \frac{R}{R_S + R} \]
FILTER REALIZATIONS

DIRECT-FORM REALIZATION ≈ SIMPLE BUT POOR PERFORMANCE FOR NARROW BAND FILTER

\[
H(s) = \frac{V_0(s)}{V_{i1}(s)} = \frac{\alpha MS^{M-1} + \ldots + a_1 S + a_0}{S^N + b_{N-1} S^{N-1} + \ldots + b_1 S + b_0}
\]

\[
\Delta = \frac{p(s)}{E(s)}
\]

CREATE N INTEGRATOR OUTPUTS \(x_0(s), x_1(s), \ldots, x_{N-1}(s)\)

SUCH THAT

\[
\frac{x_0(s)}{V_{i1}(s)} = \frac{1}{E(s)} \quad \frac{x_1(s)}{V_{i1}(s)} = \frac{s}{E(s)} \quad \ldots \quad \frac{x_{N-1}(s)}{V_{i1}(s)} = \frac{s^{N-1}}{E(s)}
\]

AND ONE SUMMATION OUTPUT \(x_N(s)\)

SUCH THAT

\[
\frac{x_N(s)}{V_{i1}(s)} = \frac{s^N}{E(s)}
\]

HOW? WRITE OUT \(x_N(s)\) AS A FUNCTION OF ITSELF AND THE INPUT \(V_{i1}(s)\)

\[
x_N(s) = \frac{s^N}{V_{i1}(s)} = \frac{1}{s^N + b_{N-1} s^{N-1} + \ldots + b_1 s + b_0}
\]

\[
x_N(s) S^N + x_N(s) b_{N-1} S^{N-1} + \ldots + x_N(s) b_1 S + x_N(s) b_0 = S^N V_{i1}(s)
\]

\[
S^N \Rightarrow
\]

\[
x_N(s) = -b_{N-1} \frac{1}{s} x_N(s) - b_{N-2} \frac{1}{s^2} x_N(s) - \ldots - b_1 \frac{1}{s^N} x_N(s) - b_0 \frac{1}{s^N} x_N(s) + V_{i1}(s)
\]
\[ V_I(s) \]

\[ X_N(s) = \frac{\sum_{n=0}^{N} V_I(s)}{E(s)} \]

\[ \frac{1}{s} X_N(s) = X_{N-1}(s) = \frac{\sum_{n=0}^{N-1} V_I(s)}{E(s)} \]

\[ \frac{1}{s^2} X_N(s) = X_{N-2}(s) = \frac{\sum_{n=0}^{N-2} V_I(s)}{E(s)} \]

\[ \frac{1}{s^{N-1}} X_N(s) = X_1(s) \]

\[ \frac{1}{s^N} X_N(s) = X_0(s) = \frac{1}{E(s)} V_I(s) \]

Finally,

\[ \frac{V_O(s)}{V_I(s)} = a_M \frac{X_M(s)}{V_I(s)} + a_{M-1} \frac{X_{M-1}(s)}{V_I(s)} + \ldots + a_1 \frac{X_1(s)}{V_I(s)} + a_0 \frac{X_0(s)}{V_I(s)} \]

\[ V_O(s) = a_M X_M(s) + a_{M-1} X_{M-1}(s) + \ldots + a_0 X_0(s) \]

\[ X_0(s) \]

\[ X_1(s) \]

\[ \vdots \]

\[ X_M(s) \]

\[ a_M \]

\[ a_1 \]

\[ \cdots \]

\[ a_0 \]

\[ V_O \]
**Example**

Consider a bandpass filter with passband gain $\approx 1$ from 9-11 kHz and stopband attenuation $\approx -40$ dB.

Zeros $z_1: \pm j(2\pi \times 20 \text{kHz})$

$z_2: + j(2\pi \times 5 \text{kHz})$

Poles @ $p_1$: $\omega_0 = 2\pi \times 9.5 \text{kHz}$, $Q_1 = 10$

$p_2$: $\omega_0 = 2\pi \times 10.5 \text{kHz}$, $Q_2 = 10$

---

[Diagram of complex plane with poles and zeros labeled.]

[Diagram of frequency response showing dB gain vs. frequency in kHz.]
\[ H(s) = \frac{0.01 \left( s + (3\pi \times 20k)^2 \right) \left( s + (2\pi \times 5k)^2 \right)}{\left( s^2 + \frac{w_{01}}{q_1} \right) \left( s^2 + \frac{w_{02}}{q_2} \right) s + w_{02}^2} \]

\[ H(s) = \frac{0.01 s^4 + 1.6778 \times 10^8 s^2 + 1.559 \times 10^{17}}{s^4 + 1.26 \times 10^4 s^3 + 7.95 \times 10^9 s^2 + 4.95 \times 10^8 s + 1.55 \times 10^9} \]

**Problem Coefficient Spread of 10^19!!!**

**Why?** *Because Integrators*

\[ H_I(s) = \frac{1}{s} \text{ has gain of 1 at 1 rad/s} \]

\[ |H_I(j1)| = 1 \]

**But** \[ |H_I(j(2\pi \times 10k))| = \frac{1}{(2\pi \times 10k)} \]

Gain \ll 1 near passband edge of filter

Instead use

\[ H(s) = \frac{1}{\frac{s^2}{\omega_0^2}} \text{ where } \frac{1}{\omega_0} = \]
SO NOW COEFFICIENTS FOR EXAMPLE ARE:

\[-b_0 \tau ^4 = -1.5508 \times 10^{14} \left( \frac{1}{2\pi \times 10k} \right)^4 = -0.99503\]

\[-b_1 \tau ^3 = -4.94856 \times 10^{13} \left( \frac{1}{2\pi \times 10k} \right)^3 = -0.199498\]\n
\[-b_2 \tau ^2 = -2.01497\]

\[-b_3 \tau = -0.2\]

\[a_4 = 0.01\]
\[a_2 \tau ^2 = 0.0425\]
\[a_0 \tau ^4 = 0.01\]

SPREAD OF 2.4
Find an active-RC realization if negative resistors allowed.

And all capacitors are 1 μF.

\[ C = \frac{1}{2\pi \times 10^6} \]

\[ R = \frac{1}{2\pi \times 10^6 \times 10} = 15.9155 \text{ kΩ} \]

There are ways of getting around negative resistors, but although this approach is straightforward, it has poor performance in terms of dynamic range and sensitivity for high Q filters.
Biquad Design

Cascades-of-biquads are the most common design approach for both analog and digital filters.

Why? =) Performance is good (though not excellent) but testing is simple (test individual biquads separately). Design is also relatively straightforward.

Steps in designing a cascade-of-biquads:

1) **Choose appropriate pole-zero pairing**

2) **Choose a suitable cascade-ordering**

3) **Realize each biquad section in desired technology** (active-RC, digital filters, switched-capacitor)

4) **Scale appropriate signal nodes to obtain maximum dynamic range.**

Let's look at each of these steps using active-RC for examples and the following transfer function:

8th order bandpass filter = equiripple passband from 1 kHz to 1.4 kHz

\[ T(s) = \begin{cases} 
0.4 \text{ dB ripple} & \text{peak gain of 0 dB} 
\end{cases} \]

Poles: \( p_1, p_1^* \), \( w_0 = 0.7106 \), \( q = 15.256 \)

\( p_2, p_2^* \), \( w_0 = 0.7911 \), \( q = 6.192 \)

\( p_3, p_3^* \), \( w_0 = 0.9188 \), \( q = 6.743 \)

\( p_4, p_4^* \), \( w_0 = 1.0048 \), \( q = 19.113 \)

\* Complex conjugate of \( p_1 \)
ZEROS: \[ z_1, z_2^* = \pm j 0.3996 \]
\[ z_3, z_4 = 0, 0 \]
\[ z_5, z_6 = \infty, \infty \]

DYNAMIC RANGE IN GENERAL

FOR THIS 8TH ORDER BANDPASS FILTER, THERE WILL BE 4 BIQUEAD FILTERS: \( T_1(s), T_2(s), T_3(s), T_4(s) \)

\[
\frac{V_1}{V_i} = T_1(s) \quad \frac{V_2}{V_1} = T_2(s) \quad \frac{V_3}{V_2} = T_3(s) \quad \frac{V_4}{V_3} = T_4(s)
\]

\[
\frac{V_o}{V_i} = T(s) = T_1(s) \cdot T_2(s) \cdot T_3(s) \cdot T_4(s)
\]

CAUZEO AN DEFINE
\[ \|T_1(s)\|_\infty = \max(|T_1(s)|, 0.416 < \omega < \infty) \]

P EAK TRANSFER-FUNCTION GAIN

ASSUMING \( V_1, V_2, V_3, V_o \) ALL CLIP AT THE SAME LEVEL
THEN WE SHOULD SCALE \( \|T_1\|, \|T_2\|, \|T_3\|, \|T_4\| \) SUCH THAT

\[ \|T\|_\infty = \|T_1\|_\infty = \|T_2\|_\infty = \|T_3\|_\infty = \|T_4\|_\infty \]

THAT WAY, FOR A SWEPT SINEWOLIO INPUT OF SAY 1 VPP,
EACH OUTPUT \( V_1, V_2, V_3, V_o \) WILL ATTAIN THE SAME PEAK VOLTAGE LEVEL (THOUGH AT DIFFERENT FREQUENCIES PERHAPS!),

THUS DYNAMIC RANGE IS OPTIMIZED IN AN \( \log \) SENSE,

FOR A GIVEN SET OF \( T_1, T_2, T_3, T_4 \)
WHY? **Dynamic Range is determined by the ratio of largest useful signal to the smallest useful signal.**

**Largest useful signal** -> Determined by maximum voltage levels possible without severe distortion (i.e. clipping).

**Smallest useful signal** -> Determined by noise considerations: thermal noise, interference, etc.

To maximize dynamic range, keep all internal signals as large as possible without distorting.

While step #3 scales for dynamic range for a given \( T_1, T_2, T_3, T_4 \), the choice of \( T_1, T_2, T_3, T_4 \) and their ordering also affect dynamic range performance.

For flat passbands, to see this consider two possible choices for \( T_i(s) \).

**Not very flat**

\[
\text{Dynamic range poor for freq near } W_{p1},
\]

(A gain later on must compensate for)

**Flat near passband of filter**
In summary, choose $T_1, T_2, T_3, T_4$ to result in as flat transfer-functions as possible around each filter's passband. (For flat passbands)

Pole-zero pairing: Each biquad needs a pair of poles and a pair of zeros

Need 4 biquads = $\frac{k_1(s-z_1)(s-z_1^*)}{(s-p_1)(s-p_1^*)}$ or $\frac{k_1(s-z_2)(s-z_2^*)}{(s-p_1)(s-p_1^*)}$

(Keep complex-conjugates together)

In general, $(\frac{n}{2})!$ different pole-zero pairing possibilities.

To get as flat transfer-functions as possible:

Note high Q poles cause most deviation in passband

& Having zeros close by poles imply cancellation

As $|f|/\omega_n$ move away from pair. Systematic procedures are available but a good approximation is:

Rule-of-thumb for pole-zero pairing:

Poles should be paired with their nearest zeros with high pole Q's taking priority.

Other choices depend on realization (e.g., often BP preferred over HP & LP due to ease of tuning)
$z_1 = 0.714j$
$z_2 = 0.4j$

$P_4$ is highest $Q \Rightarrow z_2 + P_4 \Rightarrow T_1 = \frac{k_1 (s^2 + 1.2501)}{s^2 + \frac{1.0048}{19.113} s + 1.0048^2}$

$P_1$ is next highest $Q \Rightarrow z_1 + P_1 \Rightarrow T_2 = \frac{k_2 (s^2 + 0.3996^2)}{s^2 + \frac{0.7106}{15.256} s + 0.7106^2}$

Last two BP

$T_3 = \frac{k_3 s}{s^2 + \frac{0.7911}{6.192} s + 0.7911^2}$

$T_4 = \frac{k_4 s}{s^2 + \frac{0.9188}{6.743} s + 0.9188^2}$

OR LP + HP

$T_3 = \frac{k_3 s^2}{s^2 + \frac{0.7911}{6.192} s + 0.7911^2}$

$T_4 = \frac{k_4 s^2}{s^2 + \frac{0.9188}{6.743} s + 0.9188^2}$

Choice #1

NOTE: $T_1 \Rightarrow LPN$
$T_2 \Rightarrow HPN$
$T_3, T_4 \Rightarrow BP$

Choice #2

$T_1 \Rightarrow LPN$
$T_2 \Rightarrow HPN$
$T_3 \Rightarrow BP$
$T_4 \Rightarrow LP$
CASCADE-ORDERING \Rightarrow NEED TO FIND A GOOD ORDERING FOR CASCADE-OF-BIQUADS.

\[ T_1 \Rightarrow T_2 \Rightarrow T_3 \Rightarrow T_4 \quad \text{or} \quad T_3 \Rightarrow T_2 \Rightarrow T_4 \Rightarrow T_1 \quad \text{or etc.} \]

IN GENERAL \((\frac{n}{2})!\) DIFFERENT ORDERING POSSIBILITIES

GOAL \Rightarrow KEEP TRANSFER-FUNCTIONS FLAT.

SYSTEMATIC PROCEDURE AVAILABLE BUT \ldots

RULE-OF-THUMB FOR CASCADE ORDERING

ALTERNATE HP \& LP SECTIONS AND PLACE HIGH-Q SECTIONS IN THE MIDDLE WITH LOW-Q AT EITHER END.

HP + LP ALTERNATING \Rightarrow FLATTENS OUT RESPONSE
HIGH-Q SECTIONS TOGETHER \Rightarrow FLATTEN OUT EACH OTHER IN MIDDLE FOR SYMMETRY.

IN ABOVE EXAMPLE CHOICE \# 1.

ONE OF: \begin{align*}
T_3 & \quad T_1 \quad T_2 \quad T_4 \\
T_4 & \quad T_2 \quad T_1 \quad T_3
\end{align*}

\(\text{\& \ since \ } T_3 \text{ \ peaks \ at } 0.7911 \text{ \ then \ choose } T_1, \text{ \ next which peaks \ around } 1.0098 \text{ \ (T_2 \ peaks \ around } 0.7106 \text{ \ bumps \ would \ add}).\)

CHOICE \# 2

\begin{align*}
T_3 & \quad T_1 \quad T_2 \quad T_4 \Rightarrow \quad \text{HP, LP, HP, LP} \\
T_4 & \quad T_2 \quad T_1 \quad T_3 \Rightarrow \quad \text{LP, HP, LP, LP, HP}
\end{align*}
REALIZING BIQUAD SECTIONS (ACTIVE-RC)

3 MAIN CLASSES FOR REALIZING ACTIVE-RC BIQUADS. (i.e. ACTIVE-RC 2ND ORDER TRANSFER-FUNCTIONS)

• SINGLE-AMPLIFIER BIQUADS (SABA)
  - ONE OP-AMP, POPULAR IN 70's AND AS ANTI-ALIASING FILTERS ON SOME IC's
  - WORSE DYNAMIC RANGE
  - MORE SUSCEPTIBLE TO NON-Ideal EFFECTS OF AMPLIFIER.

• TWO-INTEGRATOR-LOOP BIQUADS
  - KHAN AND TOW-THOMAS ARE 2 POPULAR TYPES
  - TYPICALLY 3 OP-AMPS REQUIRED (SOMETIMES 4)
  - GOOD DYNAMIC RANGE & LESS SENSITIVE TO NON-Ideal OP-AMPS THAN SABA

• GENERALIZED-IMPEANCE-CONVERTER (GIC) BASED
  - ONLY 2 OP-AMPS REQUIRED (SOMETIMES 3)
  - DEVELOPED BY TWO CANADIAN EE PROFS (ANTONIOU & BRUTON)
  - GOOD DYNAMIC RANGE & LESS SENSITIVE TO NON-Ideal OP-AMPS THAN SABA
  - COULD OSCILLATE IF LARGE STRAY CAPACITANCES (MIGHT REQUIRE COMPENSATION)
Two-Integrator-Loop Biquads

Based on a direct-form realization for $N=2$

KHN (Kerwin-Huelsman-Newcomb) Biquad

(also known as the State-Variable Biquad)

\[
T(s) = \frac{V_0}{V_I} = k \frac{a_2 s^2 + a_1 s + a_0}{s^2 + b_2 s + b_0} = k \frac{a_2 s^2 + a_1 s + a_0}{s^2 + \frac{w_0}{Q} s + w_0^2}
\]

And use negative integrators with $\tau = \frac{1}{w_0}$ results in
We can realize this block diagram with the following active-RC circuit.

\[ RC = \tau = \frac{1}{\omega_0} \]

In above circuit, \[ RC = \frac{1}{\omega_0} \]

Using superposition for summer

\[ V_{HP} = -V_{LP} + \left(1 + \frac{1}{\tau}\right) \left(\frac{R_1}{R_1 + R_2 + R_3}\right) V_{BP} + \left(1 + \frac{1}{\tau}\right) \left(\frac{R_2}{R_2 + R_3 + R_1}\right) V_i \]

\[ 2 \frac{R_1}{R_1 + R_2 + R_3} = \frac{1}{Q} \quad \text{and} \quad 2 \frac{R_2}{R_2 + R_3 + R_1} = k \]

2 equations \(\Rightarrow\) 3 unknowns \(\Rightarrow\) can choose size of resistors.

\(1 \text{K, } 100 \text{K, or } 1 \text{M}\)
Then output is

\[ V_0 = -\left( \frac{RF}{RH} V_{HP} + \frac{RF}{RB} V_{BP} + \frac{RF}{RL} V_{LP} \right) \]

\[
\begin{align*}
\frac{RF}{RH} &= a_2 \\
\frac{RF}{RB} &= -\frac{a_1}{w_0} \\
\frac{RF}{RL} &= \frac{q_0}{w_0^2}
\end{align*}
\]

And

\[ T(s) = -k \frac{a_2 s^2 + a_1 s + q_0}{s^2 + \frac{w_0}{\alpha} s + w_0^2} \]

If \( R_H = R_E = RF \) and \( R_B = RF \) \( Q \) then an all-pass function is realized.

\[ T(s) = -k \frac{s^2 + \frac{w_0}{\alpha} s + w_0^2}{s^2 + \frac{w_0}{\alpha} s + w_0^2} \]
TOW-THOMAS Biquad

Eliminate the VLP as an op-amp output by making all coefficients into \( \pm 1 \).

\[
\frac{1}{w_0}
\]

For finite zeros, use feed-in elements.
TIN-THOMAS USING FEEDFORWARD TO REALIZE FINITE

\[ s \cdot \frac{V_o}{v_i} = -\frac{s^2 \left( \frac{C_1}{C} \right) + s \frac{1}{C} \left( \frac{1}{R_1} - \frac{5}{R R_3} \right) + \frac{1}{R R_2 C^2}}{s^2 + \frac{1}{R C Q} s + \frac{1}{C^2 R^2}} \]