CHAPTER 12 - PROBLEMS

12.1

$$T(s) = \frac{\omega_o}{s+\omega_o} \quad T(j\omega) = \frac{\omega_o}{j\omega+\omega_o}$$

$$|T(j\omega)| = \frac{\omega_o}{\sqrt{\omega_o^2+\omega^2}}$$

$$\phi(\omega) \triangleq \operatorname{Tan}^{-1}\left[\frac{\operatorname{Im}(T(j\omega))}{\operatorname{Re}(T(j\omega))}\right]$$

$$= -\tan^{-1} \omega/\omega_o$$

$$G = 20\log_{10}|T(j\omega)|$$

$$A = -20\log_{10}|T(j\omega)|$$

ω	$\lvert T(j\omega)\rvert$ [V/V]	G [dB]	A [dB]	ϕ °
0	1	0	0	0
$0.5\omega_o$	0.8944	-0.97	0.97	-26.57
ω_o	0.7071	-3.01	3.01	-45.0
$2\omega_o$	0.4472	-6.99	6.99	-63.43
$5\omega_o$	0.1961	-14.1	14.1	-78.69
$10\omega_o$	0.0995	-20.0	20.0	-84.29
$100\omega_o$	0.010	-40.0	40.0	-89.43

12.2

$$T(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{s^3+2s^2+2s+1}$$

$$T(j\omega) = \left[j(2\omega-\omega^3)+(1-2\omega^2)\right]^{-1}$$

$$|T(j\omega)| = \left[(2\omega-\omega^3)^2+(1-2\omega^2)^2\right]^{-1/2}$$

$$= \left[4\omega^2-4\omega^4+\omega^6+1-4\omega^2+4\omega^4\right]^{-1/2}$$

$$= \left[1+\omega^6\right]^{-1/2}$$

$$= \frac{1}{\sqrt{1+\omega^6}}$$

For Phase Angle:

$$\phi(\omega) = \tan^{-1}\left[\frac{\operatorname{Im}[T(j\omega)]}{\operatorname{Re}[T(j\omega)]}\right]$$

$$= -\tan^{-1}\left[\frac{2\omega-\omega^3}{1-2\omega^2}\right]$$

For $\omega = 0.1$:

$$|T(j\omega)| = (1+0.1^6)^{-1/2} \cong 1$$

$$\phi(\omega) = -11.5° = -0.20 \text{ rad}$$

For $\omega = 1$ rad/s:

$$|T(j\omega)| = (1+1^6)^{-1/2} = 1/\sqrt{2} = 0.707$$

$$\phi = -\tan^{-1}(1/-1) = -135° = 2.356 \text{ rad}$$

Note: G = -3dB

Also: $\tan^{-1}(-1) = -45°$ or $-135°$

$\tan^{-1}(-1/1) = -45°$

$\tan^{-1}(1/-1) = -135°$

For $\omega = 10$ rad/s:

$$|T(j\omega)| = (1+10^6)^{-1/2} = 0.001$$

$$\phi = -\tan^{-1}\left[\frac{2(10)-10^3}{1-2(10^2)}\right]$$

CONT.

$$= -\tan^{-1}\left[\frac{-980}{-199}\right]$$

$$= -\left[180^\circ + \tan^{-1}\left(\frac{980}{199}\right)\right]$$

$$= -258.5^\circ$$

$$= \underline{\underline{4.512 \text{ rad}}}$$

Now consider an input of $A\sin\omega t$ to $T(s)$. The output is then given by:

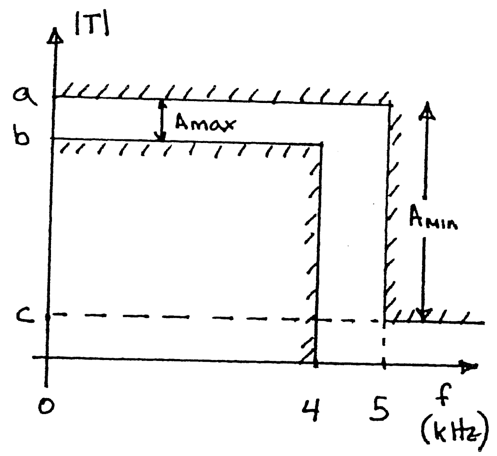
$$A\,|T(j\omega)|\,\sin(\omega t + \phi(\omega))$$

Using this result, the output to each of the following inputs will be:

INPUT	OUTPUT
$2\sin(0.1t)$	$2\sin(0.1t - 0.2)$ ie $2 \times 1 = 2$
$2\sin(1t)$	$\sqrt{2}\sin(t - 2.356)$ i.e. $2 \times 1/\sqrt{2} = \sqrt{2}$
$2\sin(10t)$	$2\times10^{-3}\sin(10t - 4.512)$

12.3

12.4



Note |T| is shown in a linear scale but Amax and Amin are in dB

From the problem

$\frac{a}{b} = 1.1$, $c = 0.1\%\,a$ or $\frac{c}{a} = 0.001$

$$A_{max} = 20\log_{10} a - 20\log_{10} b$$

$$= 20\log_{10} a/b$$

$$= 20\log_{10}(1.1)$$

$$= \underline{\underline{0.83\,dB}}$$

$$A_{min} = 20\log_{10} a - 20\log_{10} c$$

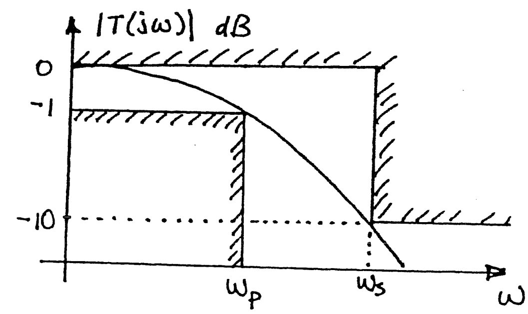
$$= 20\log_{10}(a/c)$$

$$= 20\log_{10}(0.001)$$

$$= \underline{\underline{60\,dB}}$$

$$\text{Selectivity} = \frac{\omega_s}{\omega_p} = \frac{2\pi 5}{2\pi 4} = \underline{\underline{1.25}}$$

12.5



$$T(s) = \frac{k}{1 + s\tau}$$

If $\tau = 1s$ & the DC gain $= 1$ then $\underline{k = 1}$

$$= \frac{1}{1 + 1s}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

At the passband edge:

CONT.

$$|T(j\omega_p)| = \frac{1}{\sqrt{1+\omega_p^2}} = 10^{-1/20}$$

$$\therefore \quad \omega_p = \underline{\underline{0.5088 \text{ rad/s}}}$$

At the stopband edge:

$$|T(j\omega_s)| = \frac{1}{\sqrt{1+\omega_s^2}} = 10^{-10/20}$$

$$\therefore \quad \omega_s = \underline{\underline{3 \text{ rad/s}}}$$

$$\text{Selectivity} = \frac{\omega_s}{\omega_p} = \frac{3}{0.5088} = \underline{\underline{5.9}}$$

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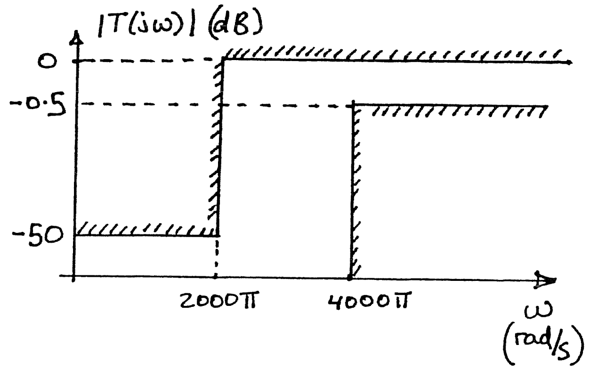
Passband is defined by: $f \geq 2\text{kHz}$

$\Rightarrow \omega_p = 2\pi(2000)$ rad/s

Stopband is defined by: $f \leq 1\text{kHz}$

$\Rightarrow \omega_s = 2\pi(1000)$ rad/s



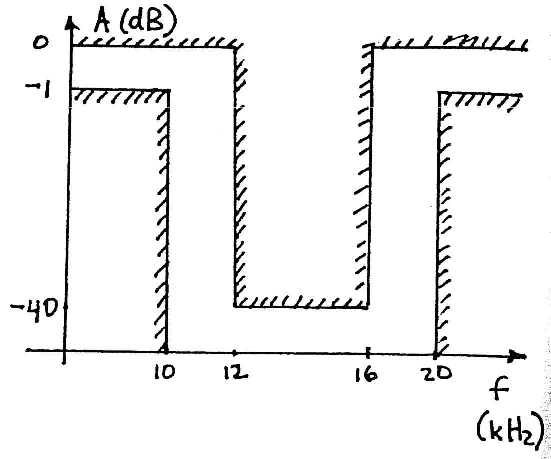
Note we assumed a maximum transmission of 0 dB.

12.7

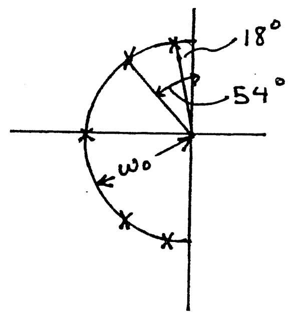
Passband: $f \in \{[0, 10\text{kHz}] \cup [20\text{kHz}, \infty]\}$

Stopband: $f \in [12\text{kHz}, 16\text{kHz}]$

$A_{max} = 1\text{dB}$, $A_{min} = 40\text{dB}$



12.8



$\omega_0 = 10^3$

Poles at 18°:

$$P_1 = \omega_0\left(-\cos(90° - 18°) \pm j\sin(90° - 18°)\right)$$
$$= \omega_0\left(-\cos 72° \pm j\sin 72°\right)$$
$$= \omega_0(-0.309 \pm j\,0.951)$$

Poles at 54°

$$P_2 = \omega_0\left(-\cos 36° \pm j\sin 36°\right)$$
$$= \omega_0(-0.809 \pm j\,0.588)$$

Poles on Real Axis

$$P_3 = -\omega_0$$

CONT.

Note: Given a pair of poles

$P_i = \omega_0(-\cos\alpha \pm j\sin\alpha)$,

introduces a second order term as follows:

$$(s+\omega_0\cos\alpha - j\omega_0\sin\alpha)(s+\omega_0\cos\alpha + j\omega_0\sin\alpha)$$

$$= s^2 + s(\omega_0\cos\alpha - j\omega_0\sin\alpha + \omega_0\cos\alpha + j\omega_0\sin\alpha)$$
$$+ \omega_0^2[\cos^2\alpha + j\cos\alpha\sin\alpha - j\cos\alpha\sin\alpha + \sin^2\alpha]$$

$$= s^2 + s(2\omega_0\cos\alpha) + \omega_0^2$$

So for P_1 we get a term:

$$s^2 + s(2\omega_0 0.309) + \omega_0^2$$
$$= s^2 + 0.618\omega_0 s + \omega_0^2$$

For P_2 we get:

$$s^2 + 1.618\omega_0 s + \omega_0^2$$

For P_3: $(s+\omega_0)$

$\therefore$ The denominator of $T(s)$ is given by

$$D(s) = (s+\omega_0)(s^2 + 0.618\omega_0 s + \omega_0^2)\times(s^2 + 1.618\omega_0 s + \omega_0^2)$$

Case (a) - If all the zeros are @ ∞, the numerator is a constant

$|T(0)| = \dfrac{k}{\omega_0^5} = 1$ for unity gain at DC

$\therefore k = \omega_0^5$

$$T(s) = \frac{k}{D(s)} = \frac{\omega_0^5}{D(s)}$$

where $D(s)$ is given above.

Case (b) - For all zeros at 0, the numerator is given by ks^5

A $s = j\infty$ $|T(s\to j\infty)| = \dfrac{k}{1} = 1$

$$\therefore T(s) = \frac{s^5}{D(s)}$$

12.9

Poles at -1 and $-0.5 \pm j0.8$ gives a denominator:

$$D(s) = (s+1)(s+0.5-j0.8)(s+0.5+j0.8)$$
$$= (s+1)(s^2 + 2(0.5)s + 0.5^2 + 0.8^2)$$
$$= (s+1)(s^2 + s + 0.89)$$

Zeros at ∞ and $\pm j2$ give a numerator:

$$N(s) = k(s+j2)(s-j2) = k(s^2+4)$$

Note there is one zero at ∞ because Degree $(D(s))$ - Degree $(N(s)) = 1$

$$T(s) = \frac{k(s^2+4)}{(s+1)(s^2+s+0.89)}$$

$|T(j0)| = \dfrac{k(4)}{0.89} = 1$ $\because$ DC gain $= 1$

$\Rightarrow k = 0.2225$

$$\therefore T(s) = \frac{0.2225(s^2+4)}{(s+1)(s^2+s+0.89)}$$

12.10

Numerator is given by
$$a_7\,(s-0)\,(s^2+(10^3)^2)\,(s^2+(3\times10^3)^2)\times(s^2+(6\times10^3)^2)$$
$$= a_7 s\,(s^2+10^6)(s^2+9\times10^6)(s^2+36\times10^6)$$

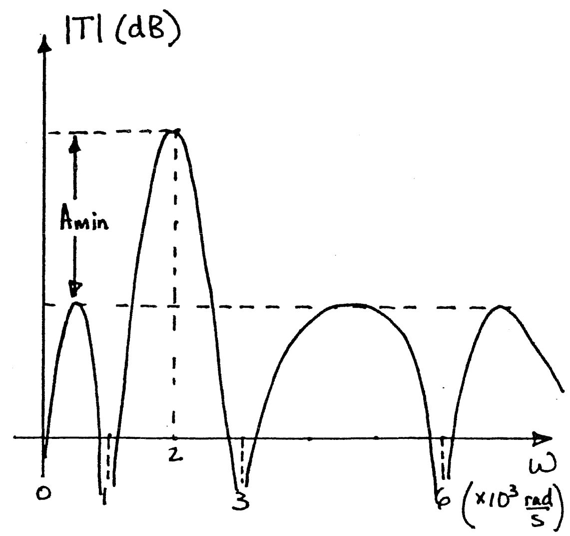
Degree of Numerator $\triangleq M = 7$

Degree of Denominator $\triangleq N$

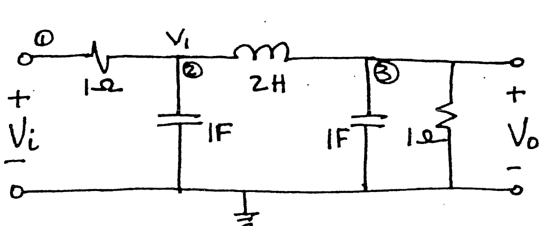
Given that there is one zero at ∞:

$$N - M = 1 \Rightarrow \underline{\underline{N = 8}}$$

$$\therefore\ T(s) = \frac{a_7 s\,(s^2+10^6)(s^2+9\times10^6)(s^2+36\times10^6)}{s^8 + b_7 s^7 + b_6 s^6 + \cdots + b_0}$$



12.11



The easiest way to solve the circuit is to use nodal analysis at nodes ①, ②, ③

At node ③ $\quad \Sigma I = 0$

$$\frac{V_o}{1} + \frac{V_o}{1/s} + \frac{V_o - V_1}{2s} = 0$$

$$\therefore V_1 = V_o\,(2s^2 + 2s + 1) \qquad \text{Eq.(a)}$$

At node ② $\quad \Sigma I = 0$

$$\frac{V_1 - V_i}{1} + \frac{V_1}{1/s} + \frac{V_1 - V_o}{2s} = 0$$

$$\therefore V_1\,(2s^2+2s+1) = V_o + 2s V_i \qquad \text{Eq.(b)}$$

(a) → (b)

$$V_o\,(2s^2+2s+1)^2 = V_o + 2s V_i$$
$$V_o\,(4s^4 + s^3(4+4) + s^2(2+4+2) + s(2+2) + 1) = V_o + 2s V_i$$

$$\frac{V_o(s)}{V_i(s)} \triangleq T(s) = \frac{2s}{4s^4 + 8s^3 + 8s^2 + 4s}$$

$$\underline{\underline{T(s) = \frac{0.5}{s^3 + 2s^2 + 2s + 1}}}$$

Poles are given by:
$$s^3 + 2s^2 + 2s + 1 = 0$$
$$(s+1)(s^2+s+1) = 0$$

$\therefore$ Poles are $\quad s = \underline{\underline{-1}}\quad$ and
$$\underline{\underline{s = -\tfrac{1}{2} \pm j\sqrt{3}/2}}$$

12.12

$A_{max} = 1\,\text{dB}$, $A_{min} = 20\,\text{dB}$, $\omega_s/\omega_p = 1.3$

Using:
$$A(\omega_s) = 10\log\left[1 + \epsilon^2\left(\frac{\omega_s}{\omega_p}\right)^{2N}\right] \qquad \text{Eq (12.15)}$$
$$= A_{MIN}$$

CONT