

Discrete-Time

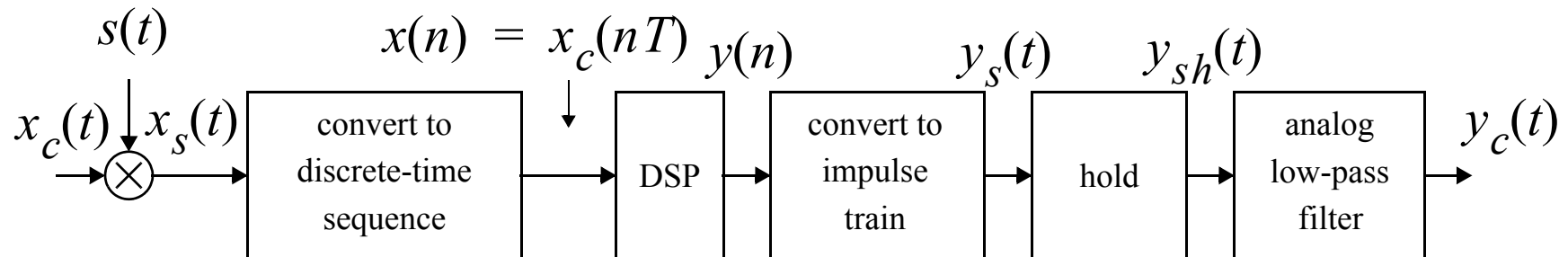
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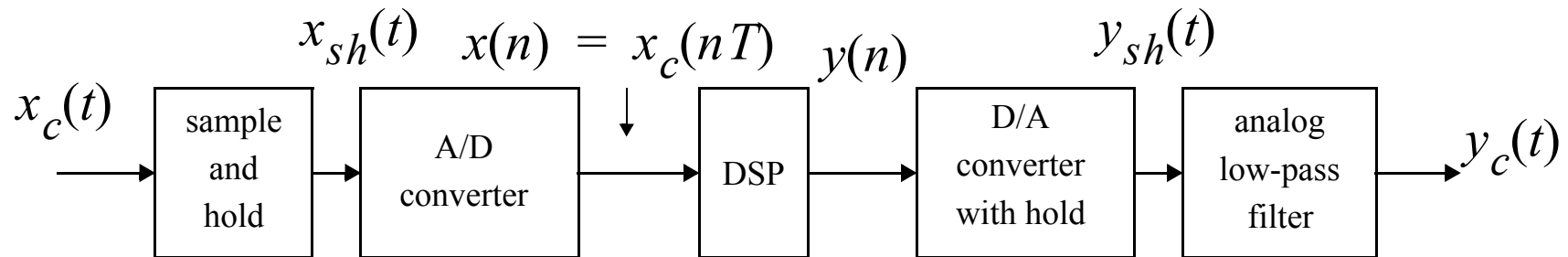
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1 of 40

Overview of Some Signal Spectra



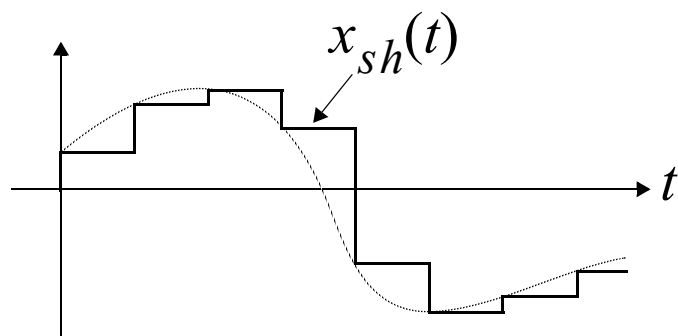
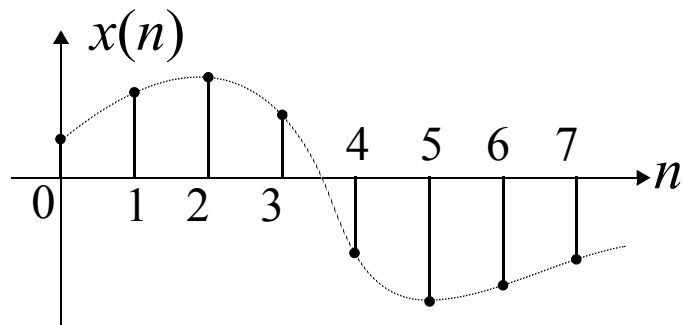
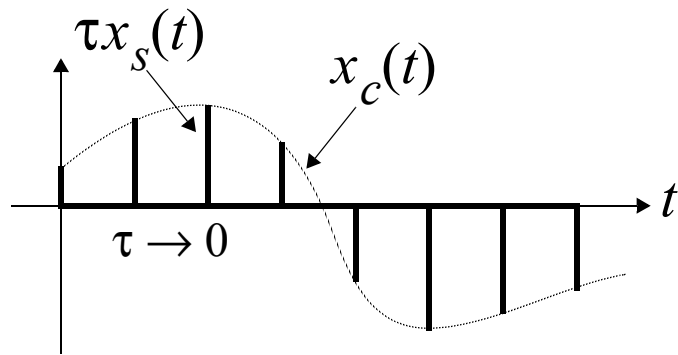
Conceptual



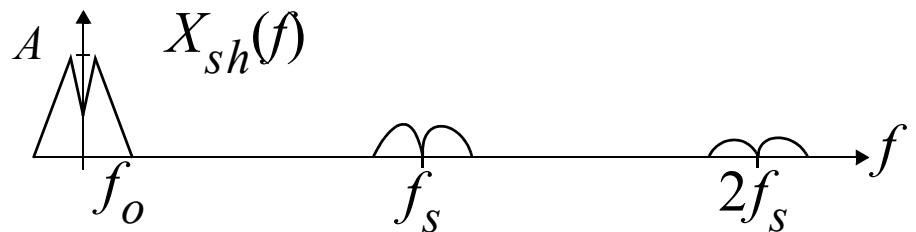
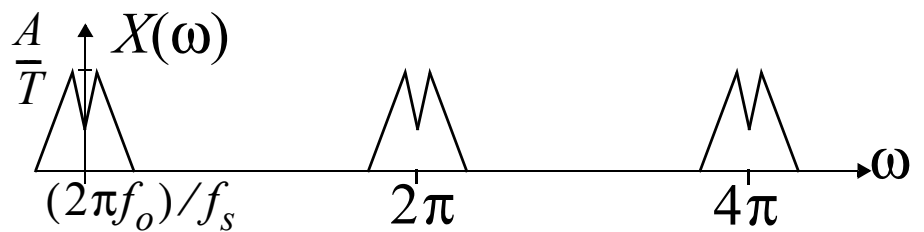
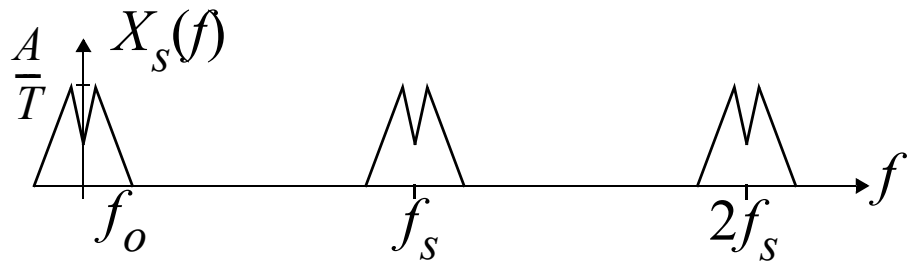
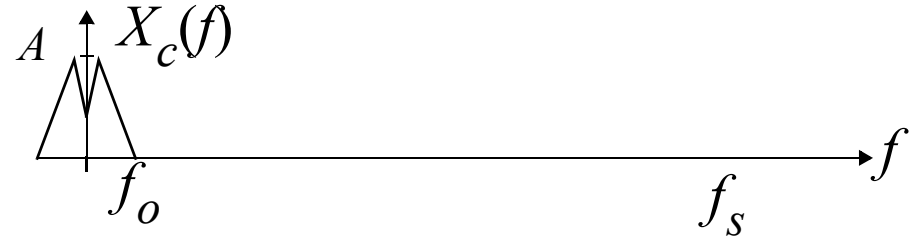
Typical Implementation



Example Signal Spectra



time



frequency

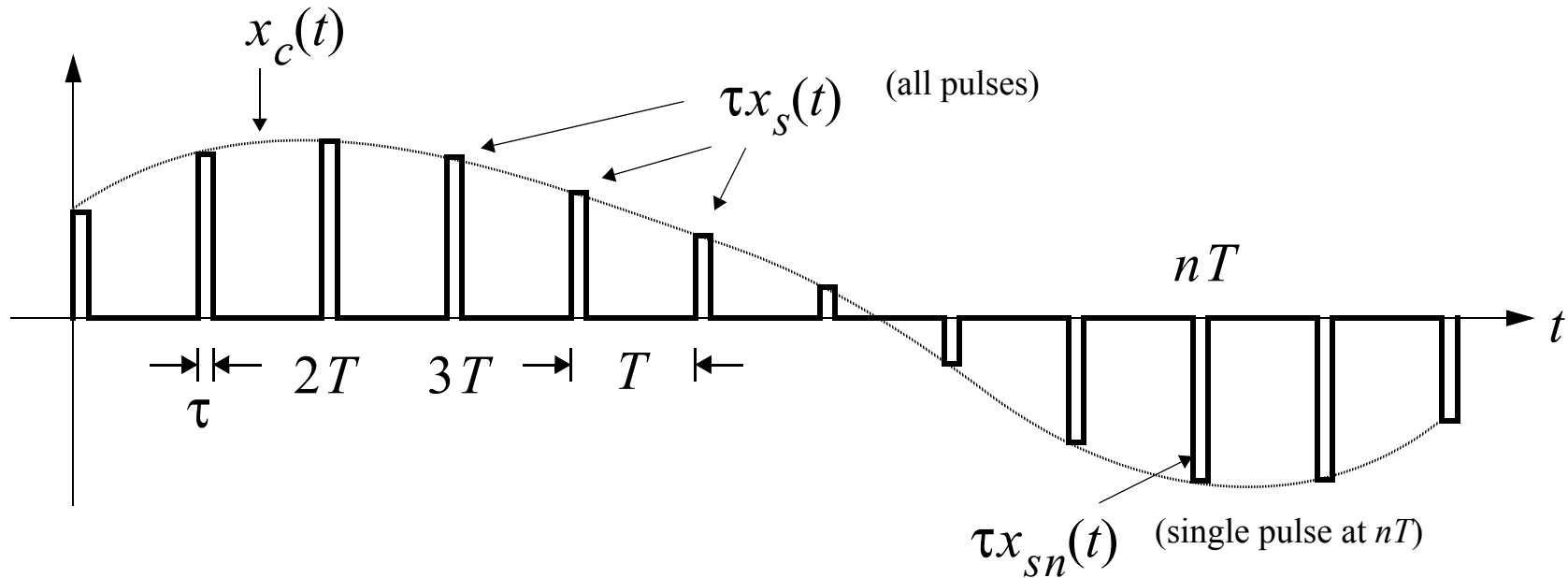


Example Signal Spectra

- $X_s(f)$ has same spectra as $X_c(f)$ but repeats every f_s (assuming no aliasing occurs).
- $X(\omega)$ has same spectra as $X_s(f)$ freq axis normalized.
- Spectra for $X_{sh}(f)$ equals $X_s(f)$ multiplied by $\frac{\sin x}{x}$ response — in effect, filtering out high frequency images.



Laplace Transform & Discrete-Time



- $x_s(t)$ scaled by τ such that the area under the pulse at nT equals the value of $x_c(nT)$.
- In other words, at $t = nT$, we have

$$x_s(nT) = \frac{x_c(nT)}{\tau} \quad (1)$$



Laplace Transform & Discrete-Time

- Thus as $\tau \rightarrow 0$, height of $x_s(t)$ at time nT goes to ∞ and so we plot $\tau x_s(t)$ instead.
- Define $\vartheta(t)$ to be the step function,

$$\vartheta(t) \equiv \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases} \quad (2)$$

- then single-pulse signal, $x_{sn}(t)$, can be written as

$$x_{sn}(t) = \frac{x_c(nT)}{\tau} [\vartheta(t - nT) - \vartheta(t - nT - \tau)] \quad (3)$$

and the entire signal $x_s(t)$ as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_{sn}(t) \quad (4)$$



Laplace Transform & Discrete-Time

- Above signals are defined for *all time* — we can find Laplace transforms of these signals.
- The Laplace transform $X_{sn}(s)$ for $x_{sn}(t)$ is

$$X_{sn}(s) = \frac{1}{\tau} \left(\frac{1 - e^{-s\tau}}{s} \right) x_c(nT) e^{-snT} \quad (5)$$

and $X(s)$ is simply a linear combination of $x_{sn}(t)$, which results in

$$X_s(s) = \frac{1}{\tau} \left(\frac{1 - e^{-s\tau}}{s} \right) \sum_{n=-\infty}^{\infty} x_c(nT) e^{-snT} \quad (6)$$



Laplace Transform & Discrete-Time

- Using the expansion $e^x = 1 + x + \frac{x^2}{2!} + \dots$, when $\tau \rightarrow 0$, the term before the summation in (6) goes to unity.
- Therefore, as $\tau \rightarrow 0$,

$$X_s(s) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT} \quad (7)$$

- This Laplace transform only depends on sample points, $x_c(nT)$ which in turn depends on the *relative* sampling-rate, T .



Spectra of Discrete-Time Signals

- $x_s(t)$ spectra can be found by replacing $s = j\omega$ in (7)
- However, a more intuitive approach is ...
- Define a periodic pulse train, $s(t)$ as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad (8)$$

where $\delta(t)$ is the unit impulse function.

- Then $x_s(t)$ can be written as

$$x_s(t) = x_c(t)s(t) \quad (9)$$

$$X_s(j\omega) = \frac{1}{2\pi} X_c(j\omega) \otimes S(j\omega) \quad (10)$$

where \otimes denotes convolution.



Spectra of Discrete-Time Signals

- Since the Fourier transform of a periodic impulse train is another periodic impulse train we have

$$S(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}) \quad (11)$$

- Thus, the spectra $X_s(j\omega)$ is found to be

$$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\omega - jk\frac{2\pi}{T}) \quad (12)$$

- or equivalently,

$$X_s(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j2\pi f - jk2\pi f_s) \quad (13)$$

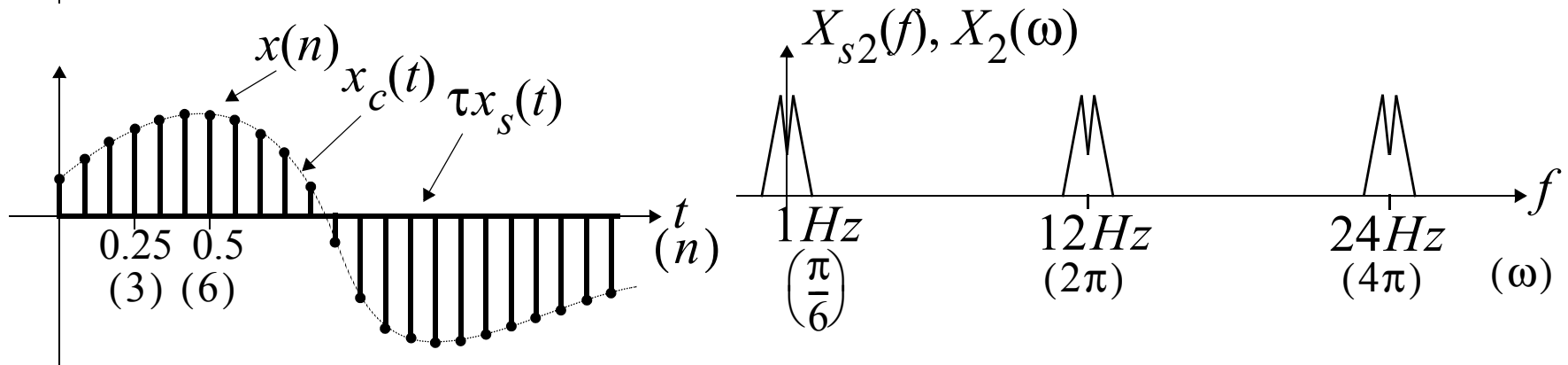
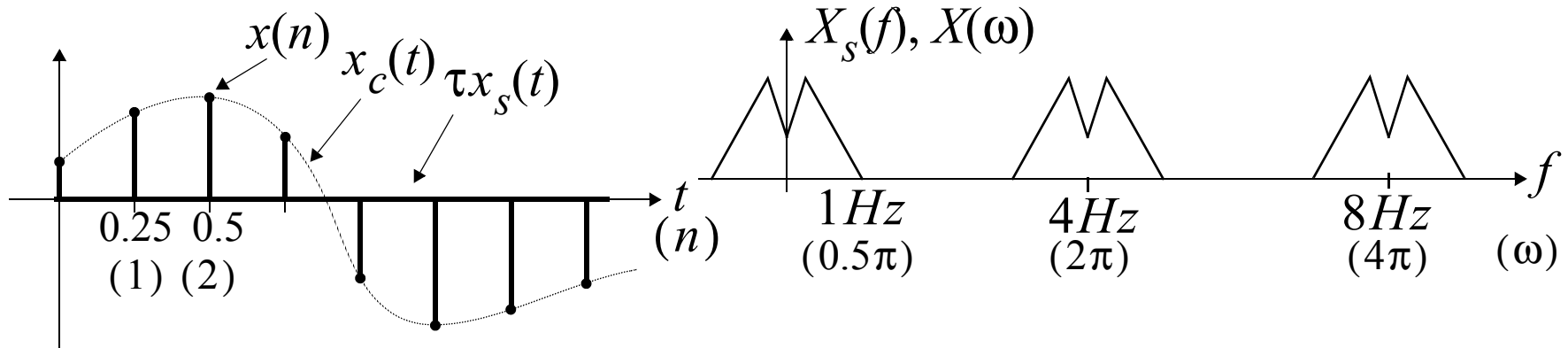


Spectra for Discrete-Time Signals

- The spectra for the sampled signal $x_s(t)$ equals a sum of shifted spectra of $x_c(t)$.
- **No aliasing** will occur if $X_c(j\omega)$ is bandlimited to $f_s/2$.
- Note that $x_s(t)$ can not exist in practice as it would require an infinite amount of power (seen by integrating $X_s(f)$ over all frequencies).



Spectra Example



Z-Transform

- The z-transform is merely a shorthand notation for (7).
- Specifically, defining

$$z \equiv e^{sT} \quad (14)$$

- we can write

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x_c(nT)z^{-n} \quad (15)$$

- where $X(z)$ is called the z-transform of the samples $x_c(nT)$.



Z-Transform

- 2 properties of the z-transform are:
 - If $x(n) \leftrightarrow X(z)$, then $x(n - k) \leftrightarrow z^{-k} X(z)$
 - Convolution in the time domain is equivalent to multiplication in the frequency domain.

$X(z)$ is not a function of the sampling-rate!

- A 1Hz signal sampled at 10Hz has the same transform as a similar 1kHz signal sampled at 10kHz
- $X(z)$ is only related to the numbers, $x_c(nT)$ while $X_s(s)$ is the Laplace transform of the signal $x_s(t)$ as $\tau \rightarrow 0$.
- Think of the series of numbers as having a sample-rate normalized to $T = 1$ (i.e. $f_s = 1\text{Hz}$).



Z-Transform

- Such a normalization results in

$$X_s(f) = X\left(\frac{2\pi f}{f_s}\right) \quad (16)$$

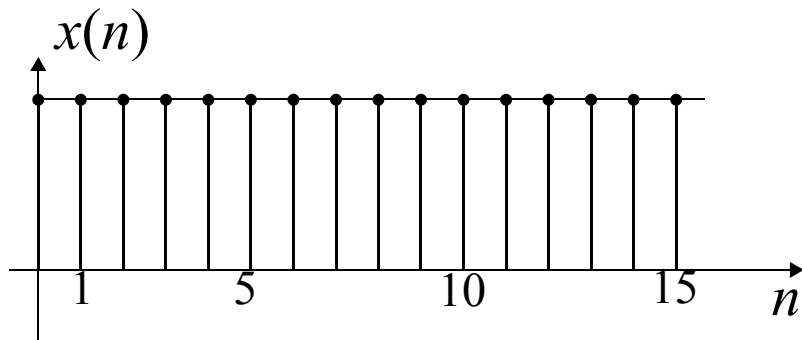
or equivalently, a frequency scaling of

$$\omega = \frac{2\pi f}{f_s} \quad (17)$$

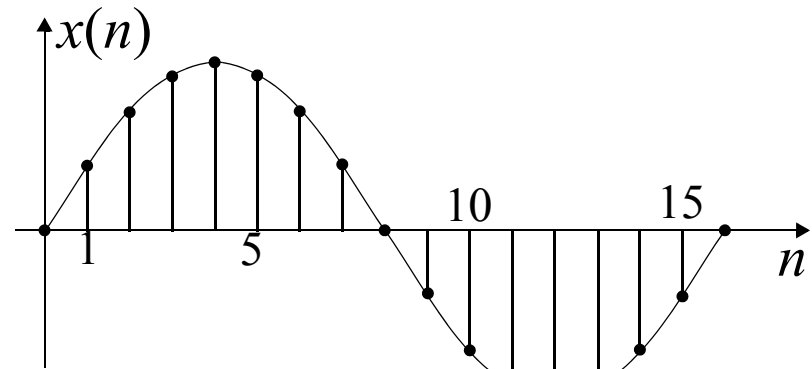
- *Thus, discrete-time signals have ω in units of radians/sample.*
- Continuous-time signals have frequency units of cycles/second (hertz) or radians/second.



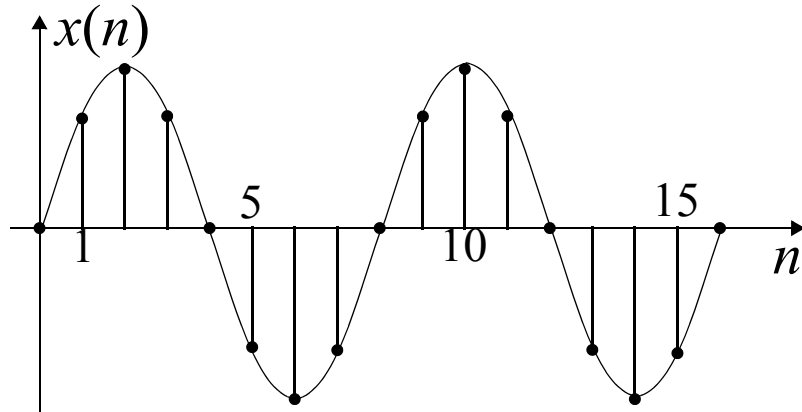
Example Sinusoidal Signals



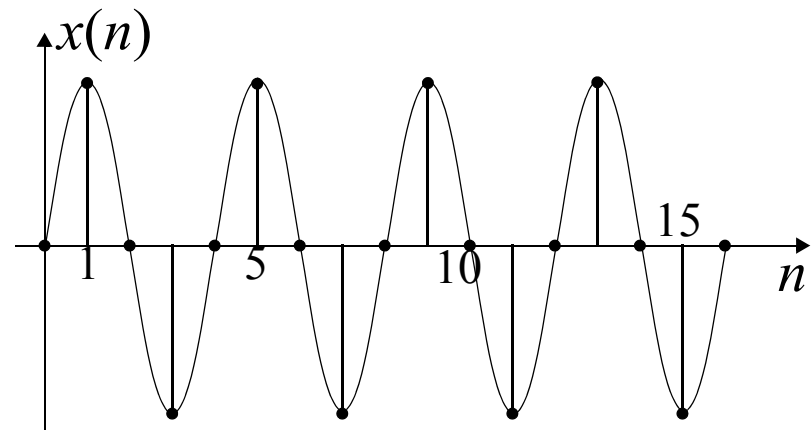
$0 \text{ rad/sample} = 0 \text{ cycles/sample}$



$\pi/8 \text{ rad/sample} = 1/16 \text{ cycles/sample}$



$\pi/4 \text{ rad/sample} = 1/8 \text{ cycles/sample}$



$\pi/2 \text{ rad/sample} = 1/4 \text{ cycles/sample}$

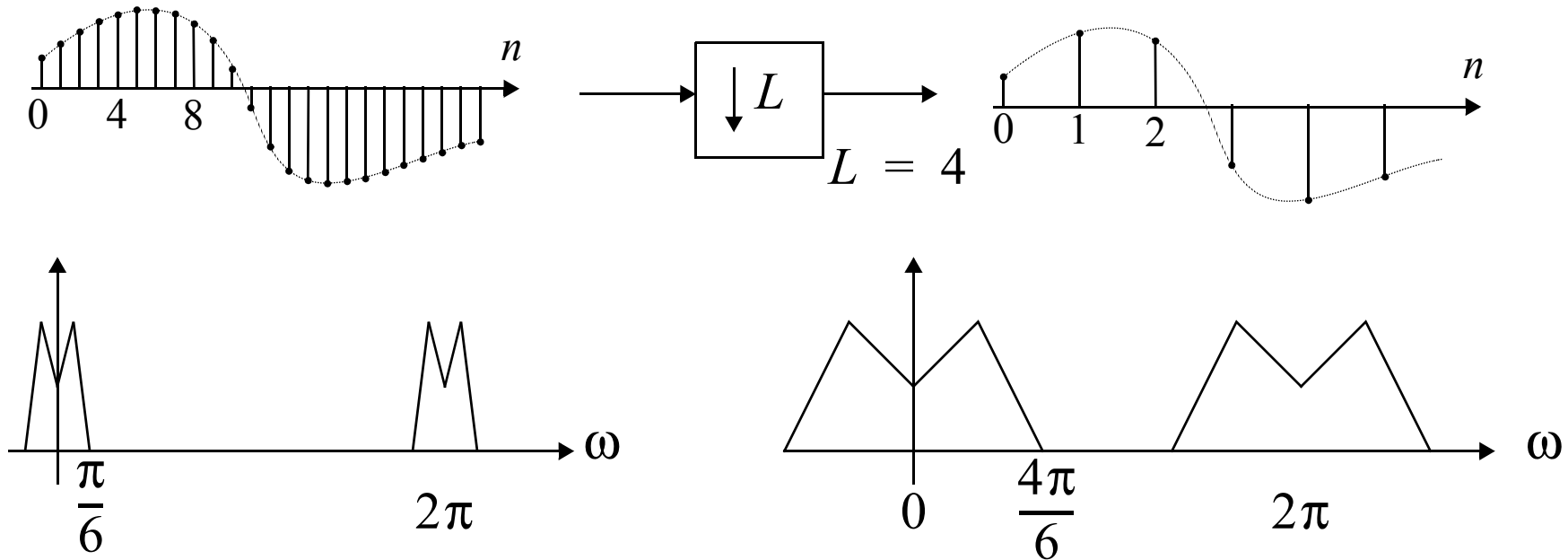


Example Sinusoidal Signals

- A continuous-time sinusoidal signal of 1kHz when sampled at 4kHz will change by $\pi/2$ radians between each sample.
- Such a discrete-time signal is defined to have a frequency of $\pi/2$ rad/sample.
- Note that discrete-time signals are not unique since the addition of 2π will result in the same signal.
- For example, a discrete-time signal having a frequency of $\pi/4$ rad/sample is identical to that of $9\pi/4$ rad/sample.
- Normally discrete-time signals are defined to have frequency components only between $-\pi$ and π rad/sample.



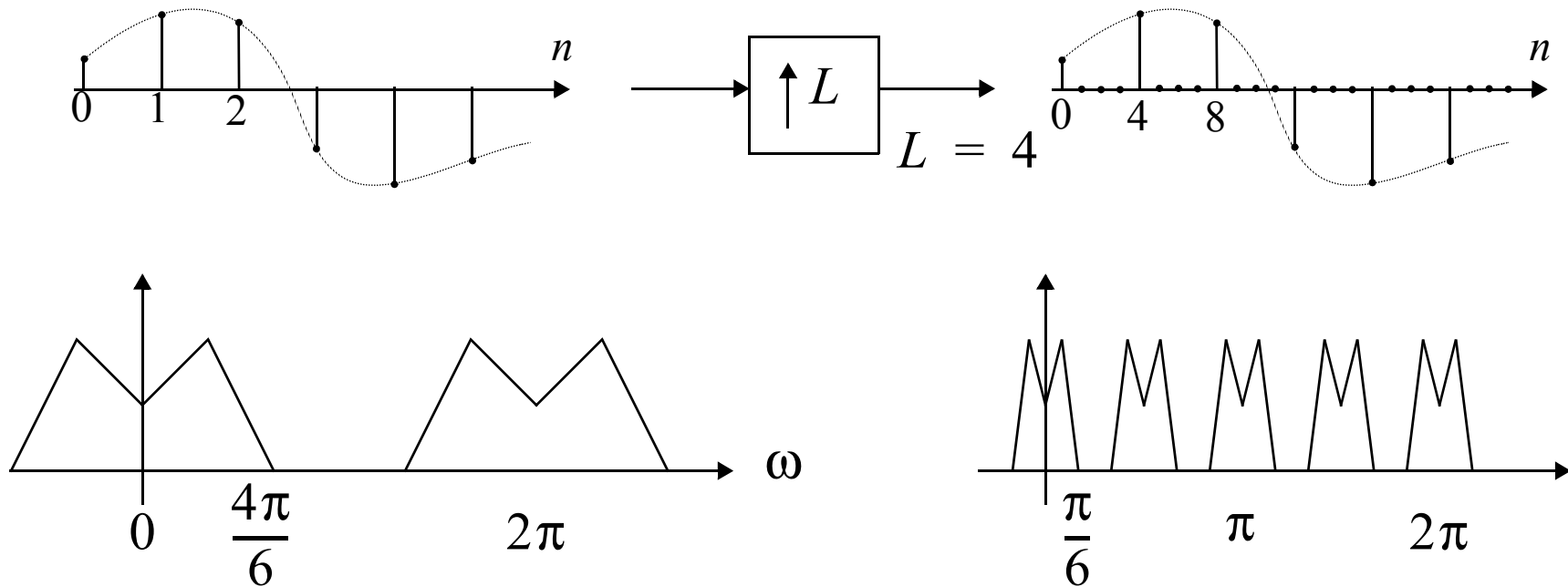
Downsampling



- Keep every L 'th sample and throw away $L - 1$ samples.
- It expands the original spectra by L .
- For aliasing not to occur, original signal must be bandlimited to π/L .



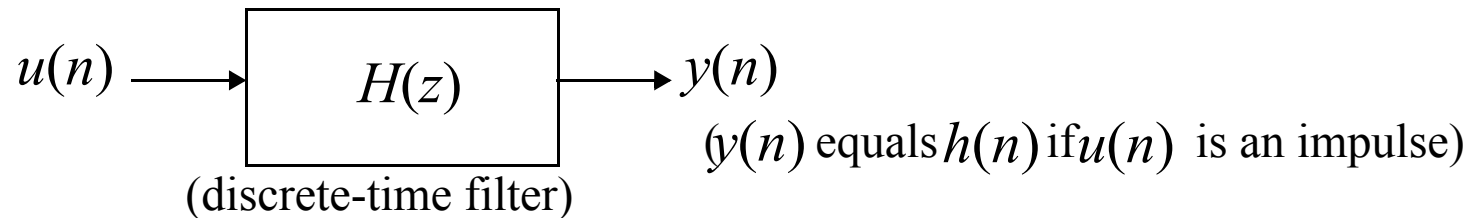
Upsampling



- Insert $L - 1$ zero values between samples
- The frequency axis is scaled by L such that 2π now occurs where $L2\pi$ occurred in the original signal.
- No worry about aliasing here.



Discrete-Time Filters



- An input series of numbers is applied to a discrete-time filter to create an output series of numbers.
- This filtering of discrete-time signals is most easily visualized with the shorthand notation of z -transforms.

Transfer-Functions

- Similar to those for continuous-time filters except instead of polynomials in “ s ”, polynomials in “ z ” are obtained.



Cont-Time Transfer-Function

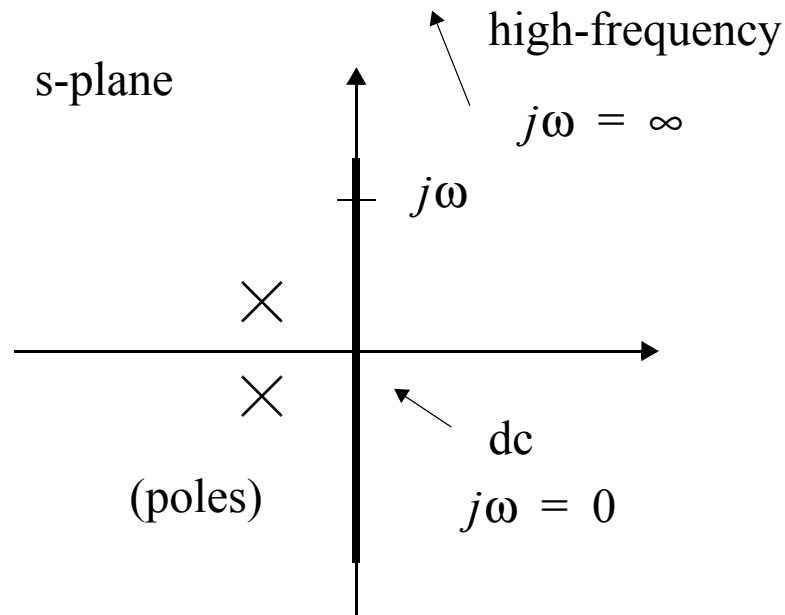
- Low-pass continuous-time filter, $H_c(s)$,

$$H_c(s) = \frac{4}{s^2 + 2s + 4} \quad (18)$$

- The poles are the roots of the denominator polynomial
- Poles: $-1.0 \pm 1.7321j$ for this example.
- Zeros: Defined to have two zeros at ∞ since the den poly is two orders higher than the numerator poly.



Cont-Time Frequency Response



- Poles and zeros plotted in the s -plane.
- Substitution $s = j\omega$ is equivalent to finding the magnitude and phase of vectors from a point along the $j\omega$ axis to all the poles and zeros.



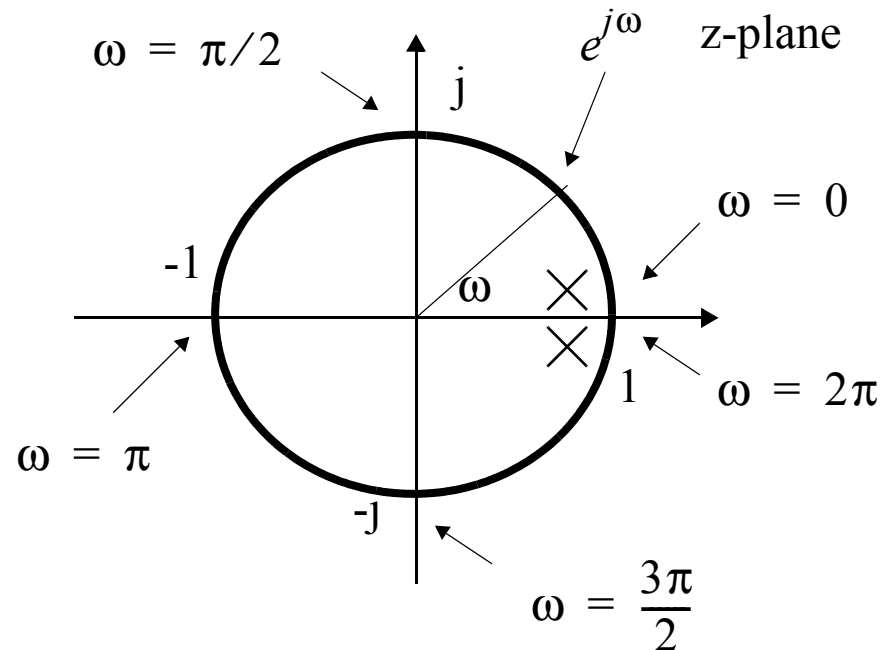
Discrete-Time Transfer-Function

$$H(z) = \frac{0.05}{z^2 - 1.6z + 0.65} \quad (19)$$

- Poles: $0.8 \pm 0.1j$ in the z -plane and two zeros are again at ∞ .
- To find the frequency response of $H(z)$, the poles and zeros can be plotted in the z -plane, and the unit circle contour is used, $z = e^{j\omega}$



Discrete-Time Frequency Response



- Note that poles or zeros occurring at $z = 0$ do not affect the magnitude response of $H(z)$ since a vector from the origin to the unit circle always has a length of unity. However, they would affect the phase response.

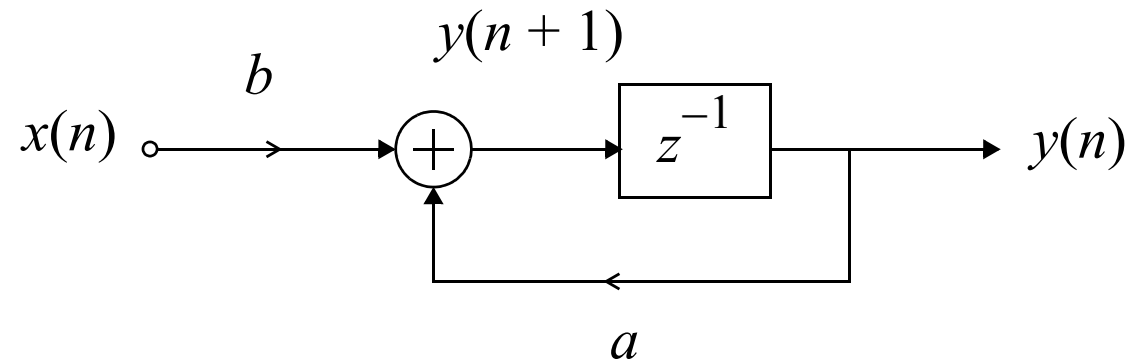


Discrete-Time Frequency Response

- $z = 1$ corresponds to the frequency response at both dc (i.e. $\omega = 0$) and for $\omega = 2\pi$.
- The time normalization of setting $T = 1$ implies that $\omega = 2\pi$ is equivalent to the sampling-rate speed (i.e. $f = f_s$) for $X_s(f)$.
- As with cont-time filter, if filter coefficients are real, poles and zeros occur in complex-conjugate pairs — magnitude is symmetric, phase is anti-symmetric.
- Going around the circle again would give the same result as the first time implying that the frequency response repeats every 2π .



Stability of Discrete-Time Filters



- To realize rational polynomials in “ z ”, discrete-time filters use delay elements (i.e. “ z^{-1} ” building blocks) much the same way that analog filters can be formed using integrators (i.e. “ s^{-1} ” building blocks).
- The result is finite difference equations describing discrete-time filters



Stability of Discrete-Time Filters

- A finite difference equation can be written for above system

$$y(n + 1) = bx(n) + ay(n) \quad (20)$$

- In the z -domain, this equation is written as

$$zY(z) = bX(z) + aY(z) \quad (21)$$

- We find $H(z)$ given by

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{b}{z - a} \quad (22)$$

which has a pole on the real axis at $z = a$.



Stability of Discrete-Time Filters

- To test for stability, let the input $x(n)$ be an impulse

$$y(0) = k$$

where k is some arbitrary initial state value for y .

$$y(1) = b + ak$$

$$y(2) = ab + a^2k$$

$$y(3) = a^2b + a^3k$$

$$y(4) = a^3b + a^4k$$

:



Stability of Discrete-Time Filters

- The response, $h(n)$, is seen to be given by

$$h(n) = \begin{cases} 0 & (n < 1) \\ (a^{n-1}b + a^n k) & (n \geq 1) \end{cases} \quad (23)$$

- This response remains bounded only when $|a| \leq 1$ for this first-order filter and is unbounded otherwise.
- *In general, a linear time-invariant discrete-time filter, $H(z)$, is stable if and only if all its poles are located within the unit circle.*



IIR Filters

- Infinite-Impulse-Response (IIR) filters are those discrete-time filters that when excited by an impulse, their outputs remain non-zero assuming infinite precision arithmetic.
- The above example is IIR when $a \neq 0$
- IIR filters can be more efficient when long impulse responses are needed.
- They have some unusual behaviors due to finite-precision effects such as limit-cycles.



FIR Filters

- Finite-Impulse-Response (FIR) filters are those discrete-time filters that when excited by an impulse, their outputs go precisely to zero (and remain zero) after a finite value of n .
- Example — running average of 3

$$y(n) = \frac{1}{3}(x(n) + x(n-1) + x(n-2)) \quad (24)$$

$$H(z) = \frac{1}{3} \sum_{i=0}^2 z^{-i} \quad (25)$$

- Has poles but they all occur at $z = 0$.
- FIR filters are always stable and exact linear phase filters can be realized.



Bilinear Transform

- Consider $H_c(p)$ as a continuous-time transfer-function (where “ p ” is the complex variable equal to $\sigma_p + j\Omega$), the bilinear transform is defined to be given by,

$$p = \frac{z - 1}{z + 1} \quad (26)$$

- The inverse transformation is given by,

$$z = \frac{1 + p}{1 - p} \quad (27)$$

- The z -plane locations of 1 and -1 (i.e. dc and $f_s/2$) are mapped to p -plane locations of 0 and ∞ , respectively.



Bilinear Transform

- The unit circle, $z = e^{j\omega}$, in the z -plane is mapped to the entire $j\Omega$ axis in the p -plane.

$$p = \frac{e^{j\omega} - 1}{e^{j\omega} + 1} = \frac{e^{j(\omega/2)}(e^{j(\omega/2)} - e^{-j(\omega/2)})}{e^{j(\omega/2)}(e^{j(\omega/2)} + e^{-j(\omega/2)})} \quad (28)$$

$$= \frac{2j \sin(\omega/2)}{2 \cos(\omega/2)} = j \tan(\omega/2) \quad (29)$$

- Results in the following frequency “warping”.

$$\Omega = \tan(\omega/2) \quad (30)$$



Bilinear Transform Filter Design

- Design a continuous-time transfer-function, $H_c(p)$, and choose the discrete-time transfer-function, $H(z)$, such that

$$H(z) \equiv H_c((z - 1)/(z + 1)) \quad (31)$$

so that

$$H(e^{j\omega}) = H_c(j \tan(\omega/2)) \quad (32)$$

- The response of $H(z)$ is seen to be equal to the response of $H_c(p)$ except with a frequency “warping”
- Order of the cont-time and discrete-time also same.



Bilinear Design Example

- Find a first-order $H(z)$ that has a 3db frequency at $f_s/20$, a zero at -1 and a dc gain of one.
- Using (30), the frequency value, $f_s/20$, or equivalently, $\omega = (2\pi)/20 = 0.314159$ is mapped to $\Omega = 0.1584$.
- Thus, $H_c(p)$ should have a 3dB frequency value of 0.1584 rad/s.
- Such a 3db frequency value is obtained by having a p -plane zero equal to ∞ and pole equal to -0.1584.



Bilinear Design Example

- Transforming these continuous-time pole and zero back using (27) results in a z -plane zero at -1 and a pole at 0.7265.
- Therefore, $H(z)$ appears as

$$H(z) = \frac{k(z + 1)}{z - 0.7265} \quad (33)$$

- The constant k can be determined by setting the dc gain to one, or equivalently, $|H(1)| = 1$ which results in $k = 0.1368$.



Sample-and-Hold Response

- A sample-and-held signal, $x_{sh}(t)$, is related to its sampled signal by the mathematical relationship,

$$x_{sh}(t) = \sum_{n=-\infty}^{\infty} x_c(nT) [\vartheta(t - nT) - \vartheta(t - nT - T)] \quad (34)$$

- $x_{sh}(t)$ is well-defined for all time and thus the Laplace transform can be found to be equal to

$$\begin{aligned} X_{sh}(s) &= \frac{1 - e^{-sT}}{s} \sum_{n=-\infty}^{\infty} x_c(nT) e^{-snT} \\ &= \frac{1 - e^{-sT}}{s} X_s(s) \end{aligned} \quad (35)$$



Sample-and-Hold Response

- The hold transfer-function, $H_{sh}(s)$, is equal to

$$H_{sh}(s) = \frac{1 - e^{-sT}}{s} \quad (36)$$

- The spectra for $H_{sh}(s)$ is found by substituting $s = j\omega$

$$H_{sh}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T \times e^{-\frac{j\omega T}{2}} \times \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \quad (37)$$



Sample-and-Hold Response

- The magnitude of this response is given by

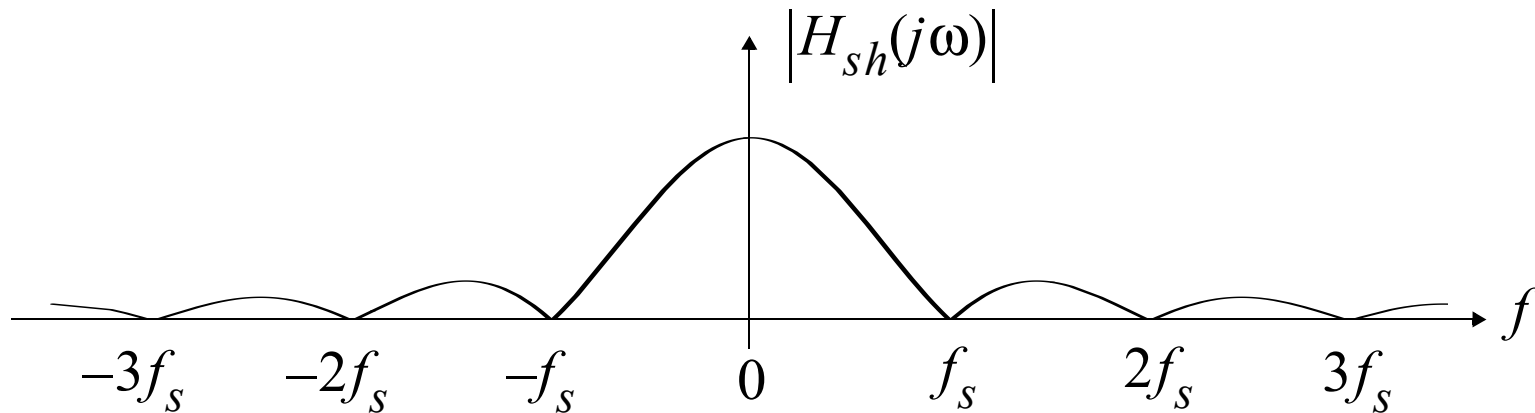
$$|H_{sh}(j\omega)| = T \frac{\left| \sin\left(\frac{\omega T}{2}\right) \right|}{\left| \frac{\omega T}{2} \right|}$$

$$\text{or} \quad |H_{sh}(f)| = T \frac{\left| \sin\left(\frac{\pi f}{f_s}\right) \right|}{\left| \frac{\pi f}{f_s} \right|} \quad (38)$$

- and is often referred to as the “ $\frac{\sin x}{x}$,” or “sinc” response.



Sample-and-Hold Response



- This frequency shaping of a sample-and-hold only occurs for a continuous-time signal.
- Specifically, a sample-and-hold before an A/D converter does not aid in any anti-aliasing requirement since the A/D converter has a true discrete-time output.

