

UPWARD PLANAR DRAWING OF SINGLE SOURCE ACYCLIC DIGRAPHS*

MICHAEL D. HUTTON[†] AND ANNA LUBIW[‡]

Abstract. An upward plane drawing of a directed acyclic graph is a plane drawing of the digraph in which each directed edge is represented as a curve monotone increasing in the vertical direction. Thomassen [24] has given a non-algorithmic, graph-theoretic characterization of those directed graphs with a single source that admit an upward plane drawing. We present an efficient algorithm to test whether a given single-source acyclic digraph has an upward plane drawing and, if so, to find a representation of one such drawing. This result is made more significant in light of the recent proof, by Garg and Tamassia, that the problem is NP-complete for general digraphs [12].

The algorithm decomposes the digraph into biconnected and triconnected components, and defines conditions for merging the components into an upward plane drawing of the original digraph. To handle the triconnected components we provide a linear algorithm to test whether a given plane drawing of a single source digraph admits an upward plane drawing with the same faces and outer face, which also gives a simpler, algorithmic proof of Thomassen's result. The entire testing algorithm (for general single-source directed acyclic graphs) operates in $O(n^2)$ time and $O(n)$ space (n being the number of vertices in the input digraph) and represents the first polynomial time solution to the problem.

Key words. algorithms, upward planar, graph drawing, graph embedding, graph decomposition, graph recognition, planar graph, directed graph

AMS subject classifications. 68Q20, 68Q25, 68R05, 68R10

1. Introduction. There are a wide range of results dealing with drawing, representing, or testing planarity of graphs. Steinitz and Rademacher [22], Fáry [10], Stein [21], and Wagner [26] independently showed that every planar graph can be drawn in the plane using only straight line segments for the edges. Tutte [25] showed that every 3-connected planar graph admits a convex straight-line drawing, where the facial cycles other than the unbounded face are all convex polygons. The first linear time algorithm for testing planarity of a graph was given by Hopcroft and Tarjan [14].

Planar graph layout has many interesting applications, and has been widely studied as a method to visualize structures commonly modeled as graphs. Combinational boolean circuits, subroutine call-charts, PERT graphs, isa-hierarchies in AI, and many other objects are naturally described with directed acyclic graphs, and are best understood visually when all edges are drawn in the same direction. Planarity is of obvious benefit in graph-drawing, so it is a natural problem to consider upward drawings in combination with planarity.

An *upward plane drawing* of a digraph is a plane drawing such that each directed arc is represented as a curve monotone increasing in the y -direction. In particular the digraph must be acyclic (a DAG). A digraph is *upward planar* if it has an upward plane drawing. Consider the digraphs in Figure 1. By convention, the edges in the diagrams in this paper are directed upward unless specifically stated otherwise, and

* This research was supported in part by NSERC, and performed while the first author was at the University of Waterloo. A preliminary version of the work was presented at the 2nd ACM/SIAM Symposium on Discrete Algorithms (SODA 1991) [16], and an extended abstract later appeared in [27], pp. 41-57.

[†] Department of Computer Science, University of Toronto, Toronto, Ontario, Canada M5S 1A4. mdhutton@cs.utoronto.ca.

[‡] Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. alubiw@uwaterloo.ca.

direction arrows are omitted unless necessary. The digraph on the left is upward planar: an upward plane drawing is given. The digraph on the right is not upward planar—though it is planar, since placing v inside the face f would eliminate crossings, at the cost of producing a downward edge. Kelly [17] and Kelly and Rival [18], and

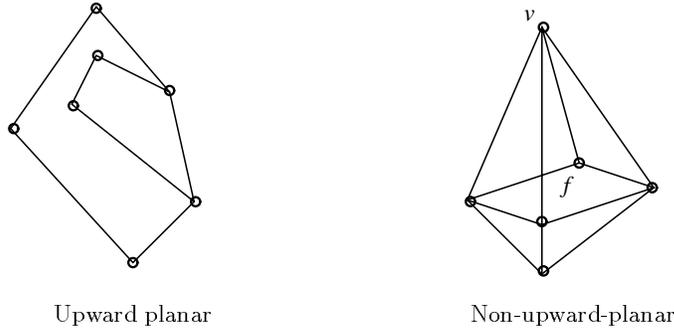


FIG. 1. *Upward planar and non-upward planar digraphs.*

also Di Battista and Tamassia [7], have shown that for every upward plane drawing there exists a *straight-line* upward plane drawing with the same faces and outer face, in which every edge is represented as a straight line segment. This is an analogue of the previously mentioned straight-line drawing result for undirected planar graphs. The general problem of recognizing upward planar digraphs has recently been shown to be NP-complete [12]. For the case of single-source single-sink digraphs there is a polynomial time recognition algorithm provided by Platt’s result [19] that such a digraph is upward planar iff the digraph with a source-to-sink edge added is planar. An algorithm to find an upward plane drawing of such a digraph was given by Di Battista and Tamassia [7]. For the special case of bipartite digraphs, upward planarity is equivalent to planarity [6].

In this paper we will give an efficient algorithm to test upward planarity for single-source digraphs, eliminating the single-sink restriction. For the most part we will be concerned only with constructing an upward planar *representation*—enough combinatorial information to specify an upward plane drawing without giving actual numerical coordinates for the vertices. This notion will be made precise in Section 3. We will remark on the extension to a drawing algorithm in Section 7. Our main result is an $O(n^2)$ algorithm to test whether a given single-source, n -vertex, digraph is upward planar, and if so, to give a representation for it which leads to a drawing with known methods. This result is partly based on a graph-theoretic result of Thomassen [24, Theorem 5.1]:

THEOREM 1.1 (THOMASSEN). *Let Γ be a plane drawing of a single-source digraph G . Then there exists an upward plane drawing Γ' strongly equivalent to (i.e. having the same faces and outer face as) Γ if and only if the source α of G is on the outer face of Γ , and for every cycle Σ in Γ , Σ has a vertex β which is not the tail of any directed edge inside or on Σ .*

The necessity of Thomassen’s condition is clear: for a digraph G with upward plane drawing Γ' , and for any cycle Σ of Γ' , the vertex of Σ with highest y -coordinate cannot be the tail of an edge of Σ , nor the tail of an edge whose head is inside Σ .

Since a 3-connected graph has a unique planar embedding (up to the choice of the outer face) by Whitney’s theorem (c.f. [2]) Thomassen concludes that his theorem provides a “good characterization” of 3-connected upward planar digraphs—

i.e. puts the class of 3-connected upward planar digraphs in NP intersect co-NP. An efficient algorithm is not given however (there are potentially an exponential number of possible cycles to check), nor does Thomassen address the issue of non-3-connected digraphs (which could have an exponential number of different planar embeddings).

The problem thus decomposes into two main issues. The first is to describe Thomassen’s result algorithmically; we do this in Section 4 with a linear time algorithm, which provides an alternative proof of his theorem. The second issue is to isolate the triconnected components of the input digraph, and determine how to put the “pieces” back together after the embedding of each is complete. This more complex issue is treated in Section 6, after a discussion of decomposition properties in Section 5.

The algorithm for splitting the input into triconnected components and merging the embeddings of each operates in $O(n^2)$ time. Since a triconnected graph is uniquely embeddable in the plane up to the choice of the outer face, and the number of possible external faces of a planar graph is linear by Euler’s formula, the overall time to test a given triconnected component is also $O(n^2)$, so the entire algorithm is quadratic.

2. Preliminaries. In addition to the definitions below we will use standard terminology and notation of Bondy and Murty [2].

All digraphs in this paper are acyclic unless otherwise stated, and n always denotes the number of vertices in the current digraph. We will use the term *cycle* and the various notions of connectivity with respect to the *underlying undirected graph*, so a digraph G is *connected* if there exists an undirected path between any two vertices in G . For S a set of vertices, $G \setminus S$ denotes G with the vertices in S and all edges incident to vertices in S removed. If S contains a single vertex v we will use the notation $G \setminus v$ rather than $G \setminus \{v\}$. G is k -connected if it has at least $k + 1$ vertices and the removal of at least k vertices is required to *disconnect* the graph. By Menger’s Theorem [2] G is k -connected if and only if there exist k vertex-disjoint undirected paths between any two vertices. A set of vertices whose removal disconnects the graph is a *cut-set*. The terms *cut vertex* and *separation pair* apply to cut-sets of size one and two respectively. A graph which has no cut vertex is *biconnected* (2-connected). A graph with no separation pair is *triconnected* (3-connected). For G with cut vertex v , a *component* of G with respect to v is formed from a connected component H of $G \setminus v$ by adding to H the vertex v and all edges between v and H . For G with separation pair $\{u, v\}$, a *component* of G with respect to $\{u, v\}$ is formed from a connected component H of $G \setminus \{u, v\}$ by adding to H the vertices u, v and all edges between u, v and vertices of H . The edge (u, v) , if it exists, forms a component by itself. An algorithm for finding triconnected components¹ in linear time is given in Hopcroft and Tarjan [13]. A related concept is that of *graph/digraph union*: we define $G_1 \cup G_2$, for components with “shared” vertices to be the *inclusive* union of all vertices and edges. That is, for v in both G_1 and G_2 , the vertex v in $G_1 \cup G_2$ is adjacent to edges in each of the subgraphs G_1 and G_2 .

Contracting an edge $e = (u, v)$ in a graph G results in a graph, denoted G/e , with the edge e removed, and vertices u and v *identified*. Inserting new vertices within edges of G generates a *subdivision* of G . A *directed subdivision* of a digraph G results from repeatedly adding a new vertex w to divide an edge (u, v) into (u, w) and (w, v) . (Directed) graphs G_1 and G_2 are *homeomorphic* if both are (directed) subdivisions of some other (directed) graph. G is planar if and only if every subdivision of G is

¹ Note that Hopcroft and Tarjan’s “components” include an extra (u, v) edge.

planar [2].

In a directed graph, the *in-degree* of a vertex v is the number of edges directed towards v , denoted $\deg^- v$. Analogously the *out-degree* ($\deg^+ v$) of v is the number of edges directed away from v . A vertex of in-degree 0 is a *source* in G , and a vertex of out-degree 0 is a *sink*.

Adopting some poset notation: we will write $u \leq v$ if there is a directed path $u \xrightarrow{*} v$ of length 0 or more, and $u < v$ ($u \xrightarrow{+} v$) to emphasize that u and v are distinct. Vertices u and v are *comparable* if $u \leq v$ or $v \leq u$, and *incomparable* otherwise. If (u, v) is an edge of a digraph then u *dominates* v , u is *incident to* v , and v is *incident from* u .

3. A Combinatorial View of Upward Planarity. As discussed by Edmonds and others (see [11]) a connected graph G is planar iff it has a *planar representation*: a cyclic ordering of edges around each vertex such that the resulting set of *faces* F satisfies $2 = |F| - |E| + |V|$ (Euler’s formula). A *face* is a cyclically ordered sequence of edges and vertices $v_0, e_0, v_1, e_1, \dots, v_{k-1}, e_{k-1}$, where $k \geq 3$, such that for any $i = 0, \dots, k-1$ the edges e_{i-1} (subscript addition modulo k) and e_i are incident with the vertex v_i and consecutive in the cyclic edge ordering for v_i .

We will say that two plane drawings are *equivalent* if they have the same representation—i.e. the same set of faces. Two plane drawings are *strongly equivalent* if they have the same representation and the same outer face.

One method of combinatorially specifying an upward planar drawing is provided by the following result of (independently) Di Battista and Tamassia [7], and Kelly [17]. They use the concept of a planar *s-t digraph*, defined to be a planar DAG which has a single source s , a single sink t and contains the edge (s, t) —exactly the upward planarity condition of Platt [19] for single-source single-sink digraphs.

THEOREM 3.1 (DI BATTISTA AND TAMASSIA, KELLY). *Let G be a directed acyclic graph. If G is upward planar then edges can be added to it to obtain a planar s-t digraph (i.e. G is a (spanning) subgraph of a planar s-t digraph). Conversely, if edges can be added to G to obtain a planar s-t digraph G' , then G is upward planar. Furthermore, for any planar embedding Γ of G' with (s, t) on the outer face, there is an upward plane drawing of G strongly equivalent to Γ with the extra edges removed.*

The final statement was not explicitly given, however to prove their result, Di Battista and Tamassia give an algorithm which takes a planar *s-t* digraph, finds an *arbitrary* planar representation of it and outputs an upward plane drawing which respects this embedding, so the statement follows. Their algorithm, which we will require later in the paper, runs in $O(n)$ arithmetic² steps ($O(n \log n)$ arithmetic steps for a straight-line drawing).

The disadvantage of this NP characterization in terms of planar *s-t* digraphs is the difficulty of testing it. Thomassen’s co-NP condition on single-source digraphs suffers from the same problem. For the case of single-source digraphs, we will give a testable (algorithmic) characterization in the next section.

To provide some motivation for this algorithm, we give another characterization of single-source upward planar digraphs, equivalent to Thomassen’s:

First we define $P(v)$, the *predecessor set* of v to be the set $\{u : u \leq v \text{ in } G\}$. Notice the set $P(v)$ includes v . Define G_v to be induced subgraph of G on $P(v)$. For

² It is important to specify the time in arithmetic steps, because the algorithm is necessarily output sensitive: coordinates can require $\Omega(n)$ bits each [8].

a planar representation Γ of G , define Γ_v to be the planar representation induced by Γ on G_v .

PROPOSITION 3.2. *Given a single source DAG G , and a planar representation Γ of G with a specified outer face and source s on the outer face, G has an upward plane drawing strongly equivalent to Γ iff the following condition holds:*

CONDITION 3.3. *For each vertex $v \in V$, v is a sink on the outer face of the planar embedding Γ_v induced by $P(v)$.*

We will often refer to a planar representation Γ satisfying Condition 3.3 as an *upward planar representation* of G .

Since a strongly equivalent upward plane drawing provides the same planar representation Γ and predecessors of v have smaller y -coordinates in the drawing, v must be on the outer face of Γ'_v . Thus the necessity holds. We will complete the proof of this in Section 5; it is not necessary for the algorithm in the next section.

4. Strongly-Equivalent Upward Planarity. Consider the following question: Given a single-source acyclic digraph G and a planar representation Γ for G , with s on the outer face of Γ , does G admit an upward planar drawing strongly equivalent to Γ ?

Define a *violating cycle* of G with respect to Γ to be a cycle Σ such that every vertex of Σ is the tail of an edge inside or on Σ . This is the condition arising from Thomassen's Theorem (1.1). As observed in the introduction, a violating cycle in Γ precludes the existence of an strongly equivalent upward drawing.

We present a linear time algorithm to test whether G has an upward planar embedding strongly equivalent to Γ with a designated outer face. The algorithm will return the edges necessary to augment G so that sinks occur only on the outer face in the positive case, or a violating cycle in the negative case. Since any planar representation of a single-source DAG with the source and all sinks on the outer face is a subgraph of a planar s - t digraph—simply designate one sink as t , and add an edge from the source and all other sinks to it—the algorithm provides a new proof of Thomassen's theorem.

The algorithm is recursive, and the proof that it works is by induction. If there is a sink v on the outer face of Γ , then recursively (trivial if G has one node) determine a violating cycle for $G \setminus v$ (in which case we are done) or a set of edges X required to augment $\Gamma \setminus v$ ($G \setminus v$) to a planar representation Γ' with all sinks on the outer face. Now add v and edges incident to v to the outer face of Γ' . To determine the additional required edges to resolve the internal sinks in the new faces, consider all vertices w which are sinks on the outer face of Γ' , but are not sinks on the outer face of Γ . Adding the edges (w, v) (where they do not already exist in G) to X retains planarity, single-sourcedness and acyclicity in $G \cup X$ and does not change the outer face.

It remains to deal with the case when the outer face of Γ has no sink. We claim that in this case G has a violating cycle: If the outer face of Γ is a cycle then it is a violating cycle. If the outer face is a walk, then follow it starting at s , and let v be the first vertex which repeats. Vertex v must be a cut vertex. Consider the segment of the walk from v to v . If this segment contains only one other vertex, say u , then u is a sink, contradiction. Otherwise we obtain a cycle C from v to v . The two edges incident with v must be directed away from v and no other vertex is a sink on C , so C must be a violating cycle.

The above algorithm can be implemented in linear time (so that each vertex is involved in no more than a constant number of operations), using data structures no more complicated than a linked list. We then have

THEOREM 4.1. *Given an n -vertex single-source acyclic digraph G and a plane representation Γ , the above algorithm tests, in linear time, whether G admits an upward planar drawing strongly equivalent to Γ .*

5. Decomposition properties of Upward Planar Graphs. This section completes the discussion of upward planar representations and introduces various decomposition-properties of upward planar digraphs. The purpose is twofold: firstly, the properties are necessary for the proofs in the next section; secondly, they provide an intuitive look at the structure of upward planar digraphs, and hence motivate the decomposition approach we take in the recognition algorithm.

We begin by completing the proof of Proposition 3.2 from Section 3:

Proof. (Sufficiency) We need that for any planar representation Γ of G satisfying Condition 3.3, G admits an upward plane drawing strongly equivalent to Γ ; equivalently, that the existence of a violating cycle precludes Condition 3.3 from holding for some vertex v .

Suppose a violating cycle Σ exists in G with respect to Γ . Let G_Σ be the subgraph of G formed by edges and vertices inside or on Σ .

Without loss of generality, G_Σ has one source s_Σ , which must lie on Σ : If G_Σ had two sources s_1 and s_2 , then both would be on Σ . Since G has a single source s , there exist directed paths P_1 from s to s_1 and P_2 from s to s_2 . The last edge of each path is not in G_Σ . If either path has a vertex other than its terminal vertex on Σ , then adding to Σ the portion of the path from the last such vertex to the terminal produces another violating cycle enclosing a larger subgraph with one fewer sources. Otherwise, P_1 and P_2 contain a last common vertex, and adding the portions of the paths from that vertex to the terminals s_1 and s_2 produces another violating cycle enclosing a larger subgraph with one fewer sources. Thus we can assume that G_Σ has a single source s_Σ . The remainder of the proof references Figure 2.

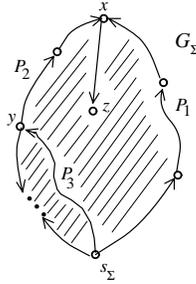


FIG. 2. *Violating cycle precludes Condition 3.1.*

Starting from s_Σ , walk counter-clockwise around Σ . Let x be the first encountered vertex with both edges of Σ directed towards x . Let y be the first encountered vertex after x with both edges of Σ directed away from y . Note that both exist, though y may be s_Σ . Let P_1 be the directed path from s_Σ to x , counter-clockwise on Σ , and let P_2 be the directed path from y to x clockwise on Σ .

Since s_Σ is the single source of G_Σ , there is a directed path P_3 in G_Σ from s_Σ to y . (If $y = s_\Sigma$ then P_3 is this single vertex.) P_3 cannot contain a vertex of P_2 other than y ; otherwise we would get a directed cycle using portions of P_2 and P_3 . Let u be the last vertex of P_1 on P_3 . Let Π be the simple undirected cycle consisting of the portion of P_3 from u to y , the portion of P_2 from y to x , and the portion of P_1 from

u to x . Let G_{Π} be the subgraph of G formed by edges and vertices inside or on Π .

Since Σ is a violating cycle, x is not a sink in G_{Σ} , so there is an edge (x, z) inside Σ , and thus inside Π . We will show that vertex z violates condition 3.1. Vertex z cannot be on Π otherwise a directed cycle is formed. Thus z is strictly inside Π . But all the vertices of Π are predecessors of z . Thus z violates condition 3.1. \square

Note that the results of the preceding two sections, combined with the characterization of Di Battista and Tamassia, and the single-source characterization of Thomassen give:

THEOREM 5.1. *The following conditions are equivalent for a single-source DAG G with planar representation Γ having a designated outer face and single source s which is on the outer face:*

- (i) G has an upward plane drawing strongly equivalent to Γ .
- (ii) G is a (spanning) subgraph of some planar s - t digraph which has an upward plane drawing strongly equivalent to Γ (after removal of the extra edges)
- (iii) for all $v \in G$, v is a sink on the outer face of Γ_v .
- (iv) Γ does not contain a violating cycle.

We note that condition (iii) is the only one which can obviously be tested in polynomial time.

In the remainder of this section we give some operations which preserve upward planarity. The first operation contracts an edge connected to a vertex of in- (out-) degree 1. The second attaches one upward planar digraph to another at a single vertex. The third attaches an upward planar digraph in place of an edge of another upward planar digraph. The last splits a vertex into two vertices.

First we will prove a useful preliminary result:

PROPOSITION 5.2. *Let G be a connected upward planar digraph. Then G is a subgraph of some single-source upward planar G^* such that all non-source $v \in V(G)$ have the same in-degree in G as in G^* .*

Proof. We illustrate how to add the edges required to “resolve the extra sources” without affecting the in-degree of non-source vertices.

Let Γ be a drawing of G in the plane bounded by x_{min} , x_{max} , y_{min} and y_{max} , with height h , width w and centred at $(0, 0)$. Without loss of generality, we assume Γ is a straight line drawing. Add new vertices s , t , l and r at $(0, -2h)$, $(0, 2h)$, $(-2w, 0)$ and $(2w, 0)$ respectively. Add lines (edges) (s, l) , (s, r) , (r, t) and (l, t) . Add further edges (s, w) for all vertices w drawn with y -coordinate of y_{min} and (w, t) for all vertices w drawn with y -coordinate of y_{max} .

The construction so far has merely added a specified outer face on the drawing, with a unique maximum sink and minimum source, so clearly the resulting drawing Γ^{**} (digraph G^{**}) is an upward plane drawing (upward planar digraph). We now wish, for each source x , to “resolve” the source by adding a new edge incident to it; the resulting digraph will prove our proposition. Let Γ be an upward plane drawing of G , and perform the following operation for each source x , except the one just added in the outer face: extend a line L vertically downwards from x to the first line or vertex in the drawing. If L first intersects a vertex w , add the (w, x) edge to both G and the drawing Γ —the result is clearly upward planar. If L first intersects an edge, rotate it along the edge until L hits some vertex w of Γ and add the edge (w, x) as before; for one of the two directions, a vertex will be found before the line becomes horizontal. Neither operation added in-degree to a non source vertex, so the claim is satisfied. \square

Note that the above is of no particular *algorithmic* significance, since the drawing

of G (or existence thereof) is the goal rather than the input. However, it allows us to prove the following lemmas in the more general context of upward planar digraphs, i.e. without the single-source assumption.

LEMMA 5.3. *Let G be a DAG and v , dominated by u , be a vertex of G with in-degree 1. Then, $G/(u, v)$ is upward planar if G is upward planar. (See Figure 3(a).)*

Note that the same result holds for G and edge (u, v) with $\deg^+ u = 1$ by symmetry. Lemma 5.3 is a generalization of a previously known fact—that G is upward planar iff any directed subdivision of G is (c.f. [24]).

Proof. Let Γ be an upward plane drawing of G . Applying proposition 5.2, there is a single-source digraph G^* containing G as a subgraph in which the in-degree of v is still 1. Clearly if the result holds for $G^*/(u, v)$ it holds for any subgraph, namely $G/(u, v)$, so we it will be sufficient to assume G is a single-source digraph for the remainder of the proof, and show Condition 3.3 holds.

Let Γ be a planar representation for G , with a designated outer face, satisfying Condition 3.3. Let Γ_w for $w \in V$ be as defined for Condition 3.3. Then Γ' , formed by contracting (u, v) in Γ , is a planar representation for $G' = G/(u, v)$ with a designated outer face. Clearly if some $w \neq v$ is on the outer face of Γ_w , it is on the other face of $\Gamma_w/(u, v)$. This, with the fact that G is acyclic, implying $G_w/(u, v) = G'_w$ for all $w \in V - \{v\}$ (i.e. $P(w)$ doesn't change as a result of contracting (u, v)), gives that w is a sink on the outer face of Γ'_w —Condition 3.3. \square

Topologically this construction can be viewed as “pulling” v and all edges incident from v down a corridor of width ϵ around (u, v) until u and v meet.

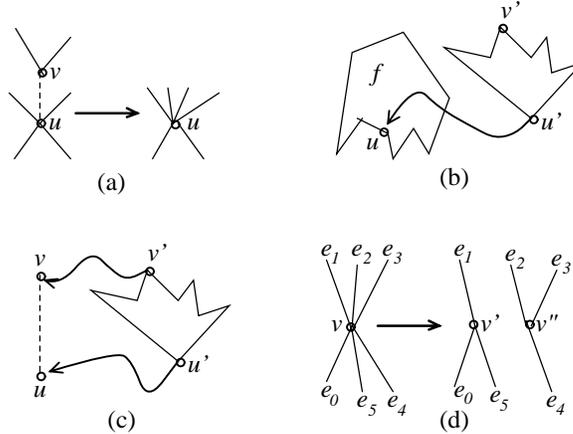


FIG. 3. *Properties of Upward Planar Representation.*

LEMMA 5.4. *Let G be an upward planar digraph with a vertex u , and let H be an upward planar digraph with a single source u' . Let G' be the digraph formed by identifying u and u' in $G \cup H$. Then G' is upward planar. (See Figure 3(b).)*

Proof. As above, there is an upward planar single-source digraph G^* containing G by proposition 5.2, and G' is a subgraph of $G^* \cup H$ with u and u' identified, so it is sufficient to prove the result for a single-source upward planar G .

Suppose Γ^G and Γ^H are the given planar representations, with designated outer faces, both satisfying Condition 3.3. Let Γ_v^G and Γ_v^H be as defined for Condition 3.3 for G and H respectively. We show how to construct a planar representation Γ' for

$G \cup H$ (identifying u and u'), with a given outer face, which satisfies Condition 3.3.

If u is on the outer face of Γ^G , then place Γ^H in the outer face, identifying u and u' . Otherwise, there are (possibly) $k + 1$ faces of Γ^G corresponding to the outer face of Γ_u^G (for k the out-degree of u); insert Γ^H in any one of these faces. It is easy to show that under this construction w is a sink on the outer face of Γ_w^G if it was a sink on the outer face of Γ_w^G (respectively Γ_w^H) previously. \square

This construction can be viewed as “inserting” the drawing for H into some face “above” u in the drawing of G .

LEMMA 5.5. *Let G be an upward planar digraph with an edge (u, v) , and H be an upward planar digraph with a single source u' and a sink v' both on the outer face. Let G' be the digraph formed by removing the (u, v) edge of G and adding H , identifying vertex u with u' and vertex v with v' . Then G' is upward planar.*

Proof. This has the same flavour as the previous proof, so we can be more brief. Again, by Proposition 5.2 it is sufficient to assume that G has a single source. Let Γ^G and Γ^H be planar representations, with designated outer faces, satisfying Condition 3.3.

Form a planar representation Γ' of G' , by replacing (u, v) by Γ^H in Γ^G . The result is planar, and has a well-defined outer face. We need that Γ' satisfies Condition 3.3. As in the previous proof, it is easy to show that w is a sink on the outer face of Γ_w^G whenever it is a sink on the outer face of Γ_w^G (respectively Γ_w^H). \square

This construction can be viewed as replacing a directed edge in an upward plane drawing of G with another upward plane drawing of H which is, in some sense, “topologically equivalent” to an edge within the drawing of G .

LEMMA 5.6. *Let G be a DAG which has an upward planar representation where the cyclic edge order about vertex v is e_0, \dots, e_{k-1} (vertices v_0, \dots, v_{k-1}). Let G' be the DAG formed by splitting v into two vertices: v' incident with edges e_i, \dots, e_j , and v'' incident with edges e_{j+1}, \dots, e_{i-1} ($i \neq j$, arithmetic mod k). Then G' is upward planar. If G had a single source, and i and j are such that each of v' and v'' retain at least one incoming edge, then the resulting G' is also a single-source digraph. (See Figure 3(d).)*

Proof. The last statement is clearly true; no new sources can be added by the construction if each new vertex has an incoming edge. Again it is sufficient to show the first part for single-source G , since the resulting digraph is otherwise a subgraph of the construction applied to G^* (of Proposition 5.2). Let Γ be a planar representation for G satisfying Condition 3.3.

Without loss of generality, assume that the construction does not make v' a source unless v was, itself, a source. We prove the result for $G'' = G' + (v', v'')$ —it is easy to augment Γ' (the planar representation formed by separating v into v' and v'' in Γ) to Γ'' with the edge (v', v'') as v' and v'' share a face. The construction of Γ'' from Γ preserves planarity and cannot introduce a dicycle; it remains to show Condition 3.3 holds for G'' and Γ'' . The set of faces in Γ'' is identical to that of Γ , save for the two new faces sharing (v', v'') , so Condition 3.3 is satisfied for all w not incident from either v' or v'' . Any v_i incident from v' or v'' (via edge e_i) is clearly on the outer face of Γ'' , whenever it is on the outer face of Γ_{v_i} , since the construction can only add vertices to an outer face, never remove them. \square

6. Separation into Tri-Connected Components. The algorithm of Section 4 tests for upward planarity of a single-source DAG G starting from a given planar representation and outer face of G . In principle, we could apply this test to all planar representations of G , but this would take exponential time. In order to avoid this, we

will decompose the digraph into biconnected and then into triconnected components. Each triconnected component has a unique planar representation (see [2]), and only a linear number of possible outer faces. We can thus test upward planarity of the triconnected components in quadratic time using the algorithm of Section 4. Since we will perform the splitting and merging of triconnected components in quadratic time, the total time will then be quadratic.

To decompose G into biconnected components we use:

LEMMA 6.1. *A DAG G with a single source s and a cut vertex v is upward planar iff each of the k components H_i of G (with respect to v) is upward planar.*

Proof. If G is upward planar then so are its subgraphs the H_i 's. For the converse, note that if $v \neq s$ then v is the unique source in all but one of the H_i 's; and if $v = s$ then v is the unique source in each H_i . Apply Lemma 5.4. \square

Dividing G into triconnected components is more complicated, because the cut-set vertices impose restrictive structure on the merged digraph. In the biconnected case, it is sufficient to simply test each component separately, since biconnected components do not interact in the combined drawing. The analogous approach for triconnected components would be to add a new edge between the vertices of the cutset in each component, then perform the test recursively. This, however, does not suffice for upward planarity, as illustrated by the two examples in Figure 4. (Recall our convention that direction arrow-heads are assumed to be “upward” unless otherwise specified.) In (a), the union of the digraphs is upward planar, but adding the edge (u, v) to each makes the second component non-upward-planar. In (b), the digraph is non-upward-planar, but each of the components is upward planar with (u, v) added.

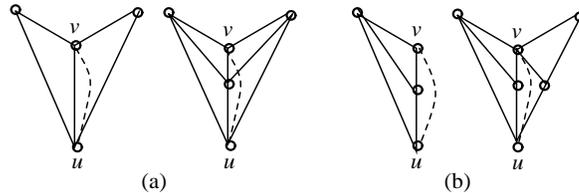


FIG. 4. Added complication of 2-vertex cut-sets.

We will find it convenient to split the digraph G into exactly two pieces at a separation pair $\{u, v\}$, where one of these pieces, E , is a component with respect to the separation pair, and the other piece, F , is the union of the remaining components. This forces each piece to fit into one face of the embedding of the other piece:

LEMMA 6.2. *For G, E, F as above, let Γ be a plane embedding of G , and let Γ_E and Γ_F be the embeddings induced on E and F , respectively. Then in Γ , all of E lies in a single face of Γ_F , and all of F lies in a single face of Γ_E . Furthermore, at least one of E, F , must lie in the outer face of Γ_F, Γ_E , respectively.*

Proof. Any distinct vertices x and y in E , neither being u or v , must share a path in E which avoids both u and v (lies entirely within E). Hence, for Γ a plane embedding of G and Γ_E and Γ_F the respective sub-embeddings of E and F , if vertices x and y of E are in different faces of Γ_F they could not share a path which avoids both u and v without violating planarity. Clearly, also, one of E, F must have two vertices on the outer face of the total drawing Γ (which has at least 3 vertices) and hence must lie entirely in the outer face of the other sub-drawing. \square

We will test upward planarity of a biconnected digraph G by breaking it at a cut-set into pieces E and F as above, and looking for upward planar embeddings Γ_E

and Γ_F that fit together as in Lemma 6.2.

We need a face in Γ_E that contains u and v and is the right “shape” to accommodate the “shape” of Γ_F ; and we need a face in Γ_F that contains u and v and is the right “shape” to accommodate the “shape” of Γ_E . (Figure 4(b) showed an example where these conditions fail.) These conditions will be enforced by adding a “marker” connecting u and v to E (F , respectively) that captures the “shape” of Γ_F (Γ_E , respectively), and forces u and v to lie in a common face. For example, the simplest case is when u is the source and v is the sink of F ; then the marker representing Γ_F in E is a single (u, v) edge.

Besides playing the primary role described above, the markers will also be used to make the two components 3-connected, and single-source, thus allowing us to recurse on smaller subproblems. The markers we are interested in are shown in Figure 5.

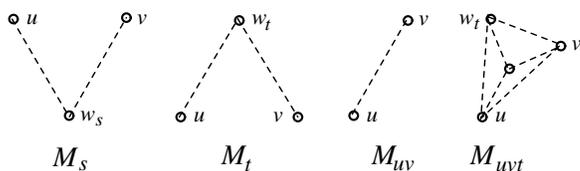


FIG. 5. Marker Graphs.

We need one other main idea. The last statement of Lemma 6.2 is that one of E, F , must lie in the outer face of Γ_F, Γ_E , respectively. For undirected digraphs this causes no problem, since any face can be made the outer one. However, for upward planarity, this condition complicates things. The situation is simplified when $s \neq u, v$. In this case we will take E to be the $\{u, v\}$ component containing s , and so Γ_E must lie in the outer face of Γ_F . When $s \in \{u, v\}$ we must do extra work to decide the “outer” component.

Having determined or decided that Γ_E must lie in the outer face of Γ_F , we know that u and v must be on the outer face of Γ_F . Thus our algorithm will solve the more general problem of testing upward planarity under the condition that some specified set X of vertices, called the “outer” set, must lie on the outer face.

To summarize, given a biconnected digraph G and an “outer” set of vertices X , we break G at a cut set $\{u, v\}$ into one component E containing s , and the union of the remaining components F . We add appropriate markers to E and F , specify their “outer” sets, and recurse. We must prove that G has an upward planar embedding with its “outer” set on the outer face iff the smaller digraphs do.

The details and proofs of this plan make up the remainder of this section. We will consider three cases separately: when u and v are incomparable; when u and v are comparable with $s < u < v$; and when u and v are comparable with $u = s$.

An important note to make at this time is that the markers, except for M_{uv} , are subgraphs attached at only two vertices, which means that $\{u, v\}$ will still constitute a cut-set. For the purposes of determining cut-sets, and making recursive calls, the markers should be treated as distinguished edges—a single edge labelled to indicate its role. As long as the type of marker is identified, the algorithm can continue to treat the vertices of attachment as source, sink or neither, as appropriate for the particular operation.

6.1. Cut-set $\{u, v\}$; u and v are incomparable. Here we consider vertex cut-sets $\{u, v\}$ which are incomparable (then neither is s). We divide the digraph G

at $\{u, v\}$ into two subgraphs—the *source component* E (the one component which contains the source s), and the union of the remaining components F .

First we need some preliminary results:

PROPOSITION 6.3. *If G is a connected DAG with exactly two sources u and v , then there exists some w_t such that two vertex disjoint (except at w_t) directed paths $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ exist in G .*

Proof. Let G be such a DAG and let P be an undirected path from u to v . Note that every x in P is comparable with either u or v , otherwise G has more than two sources. Follow P from u to the first node x (following y on P) incomparable with u (in G). Then x is comparable with v and (x, y) is an edge in G (otherwise $u < x$), so y is also comparable with v . Taking the first common vertex in the paths $u \xrightarrow{\pm} y$ and $v \xrightarrow{\pm} y$ gives w_t . \square

The following result shows the existence of lower bounds and upper bounds (in the partial order corresponding to G) under certain conditions. This allows us to prove the necessity conditions in Theorem 6.5 (to come).

LEMMA 6.4. *If G is a biconnected DAG with a single source s , and u and v are incomparable vertices in G , then there exists some w_s such that two vertex disjoint (except at w_s) directed paths $w_s \xrightarrow{\pm} u$ and $w_s \xrightarrow{\pm} v$ exist in G . If $\{u, v\}$ is a cut-set in G , then there also exists some w_t such that two vertex disjoint (except at w_t) directed paths $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ exist in G .*

Proof. Since G is a single source digraph, there exist directed paths from s to u and s to v in G . Taking the last common vertex in these paths gives w_s .

For the existence of w_t , let u and v be an incomparable separation pair of G . Since $\{u, v\}$ cuts G into at least two connected components, any non-source component H has u and v as its (exactly) two sources, and the result follows from Proposition 6.3. \square

We are now ready to proceed with the statement of the first main result of the decomposition.

THEOREM 6.5. *Let G be a biconnected directed acyclic digraph with a single source s and let $X = \{x_i\} \subseteq V(G)$ be a set of vertices. Let $\{u, v\}$ be a separation pair of G , with u and v incomparable. Let E be the connected component of G with respect to $\{u, v\}$ containing s , and F be the union of all other components. Then, G admits an upward plane drawing with all vertices of X on the outer face if and only if*

- (i) $E' = E \cup M_t$ admits an upward plane drawing with all vertices of X in E on the outer face, and w_t on the outer face if some $x \in X$ is contained in F .
- (ii) $F' = F \cup M_s$ admits an upward plane drawing with all vertices of X in F on the outer face.

Here, as in the remaining cases, the proof will have the same basic flavour. The necessity of the marker-conditions will follow from the existence of the corresponding marker ‘within’ (i.e. homeomorphic to a subgraph of) the companion component. The sufficiency will be shown by applying the properties of an upward planar representation from Section 4 to combine upward planar representations for the two subproblems into a single upward planar representation.

Proof. (Necessity): Suppose G admits an upward planar drawing (representation) Γ with all $x_i \in X$ on the outer face. Follow Figure 6.

Since u and v are incomparable, there exists a w_s and vertex-disjoint directed paths $w_s \xrightarrow{\pm} u$ and $w_s \xrightarrow{\pm} v$ in G by Lemma 6.4; specifically, these must be in E if G has a single source. Then $F' = F \cup \{(w_s, u), (w_s, v)\}$ is homeomorphic to a subgraph

of G and hence upward planar itself. F' can be obtained from G by deleting and contracting E to its marker—this will not decrease the set of vertices on the outer face. Thus, since Γ has all the x_i 's of F on its outer face, F' has an upward planar drawing with all vertices of X in F on the outer face. We have (ii).

Similarly, there exists some w_t in F such that $E \cup \{(u, w_t), (v, w_t)\}$ is homeomorphic to a subgraph of G , so E' is upward planar. As above, any x_i in E and also on the outer face of Γ will be on the outer face of the subdrawing formed by Γ_E and the $u \xrightarrow{\pm} w_t, v \xrightarrow{\pm} w_t$ paths. By Lemma 6.2, Γ_F must lie entirely within one face of Γ_E , so if some x_i in F is on the outer face of the drawing Γ , then the portion of the drawing formed by the $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ paths (hence w_t itself) must be in the outer face of the sub-drawing Γ_E , giving (i).

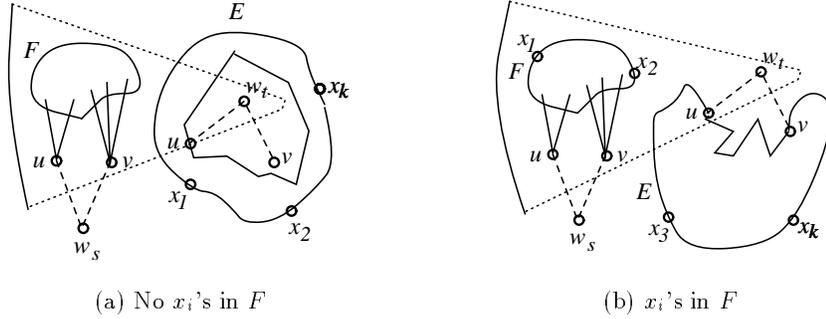


FIG. 6. Merging E and F ; cut-set $\{u, v\}$ incomparable.

(Sufficiency): Suppose E' and F' admit upward planar representations satisfying (i) and (ii). Identifying the single source w_s of F' and w_t in E' (call the new vertex w) as per Lemma 5.4, the result G' is upward planar. Splitting w into w_l with the leftmost two vertices and w_r , with the rightmost two vertices (Lemma 5.6), and contracting the (w_l, u) and (w_l, v) edges (Lemma 5.3) gives exactly G , which is hence upward planar. The construction is illustrated in Figure 7.

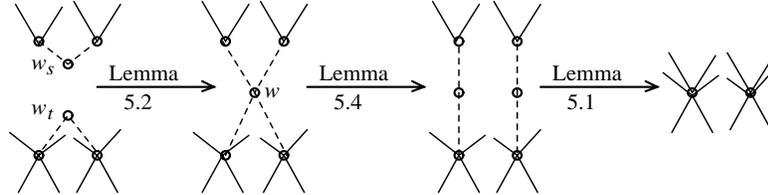


FIG. 7. Merge construction; $\{u, v\}$ incomparable.

For the sufficiency of the x_i conditions, we notice that all non-marker vertices of X on the outer face of the E' drawing are also on the outer face of the constructed drawing. If F contains no x_i this is sufficient; otherwise w_t being on the outer face of E' guarantees that the non-marker vertices of X in F' are also on the outer face of the result. \square

6.2. Cut-set $\{u, v\}$, where $u < v, u \neq s$. Here we consider any other vertex cut-sets not involving the source s . We again divide G into the *source component* E and the union of the remaining components F . Note that v can be a source in E , as long as there is a u to v path in F .

An additional preliminary result will be useful.

LEMMA 6.6. *If G is a biconnected DAG with a single source s and cut-set $\{u, v\}$, where $u < v$ in G and $u \neq s$, then in any non-source component H of G with respect to $\{u, v\}$, where $\deg^+ v > 0$, there exists some w_t such that $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ are vertex disjoint directed paths in H .*

Proof. No vertex other than u and v can be a source in H , otherwise G has more than one source; and u is always a source in H . If v is also a source, then we are done by Proposition 6.3.

If v is not a source, let $w \in H$ be a vertex dominated by v . G is biconnected, so there are two vertex disjoint $u \xrightarrow{\pm} w$ undirected paths in G . But u and v are cut-vertices in G , so at least one of the paths P lies completely within H and does not contain v (as w is in H and the only exit points from H are u and v). Every x on P is comparable with either u or v , or else G has more than one source. Find the last vertex y on P which has a $u \xrightarrow{\pm} y$ path (in G) without v . If $y = w$, then we are done. Otherwise, the vertex x following y on P has any $u \xrightarrow{\pm} x$ path necessarily going through v . Then there exist directed paths $v \xrightarrow{\pm} x$, $u \xrightarrow{\pm} x$ with the latter not containing v so the first common vertex on these paths provides a w_t . \square

We can now continue with the second main result of the decomposition.

THEOREM 6.7. *Let G be a biconnected directed acyclic digraph with a single source s , and let $X = \{x_i\} \subseteq V(G)$ be a set of vertices. Let $\{u, v\}$ be a separation pair of G with $u < v$ in G and $u \neq s$. Let E be the source component of G with respect to $\{u, v\}$ and F be the union of all other components. Then, G admits an upward plane drawing with all vertices of X on the outer face if and only if*

- (i) $E' = (E \cup F\text{-marker})$ admits an upward plane drawing with all vertices of X in E on the outer face and w_t (if it exists, otherwise the edge (u, v)) on the outer face if some $x \in X$ is contained in F .
- (ii) $F' = (F \cup E\text{-marker})$ admits an upward plane drawing with w_t (if it exists, otherwise the edge (u, v)) and all vertices of X in F on the outer face.

where

$$F\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } F \\ M_{uv} & \text{if } v \text{ is a sink in } F \\ M_{uvt} & \text{otherwise.} \end{cases}$$

and

$$E\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } E \\ M_{uv} & \text{otherwise.} \end{cases}$$

Proof. (Necessity) Suppose G admits an upward plane drawing with all $x_i \in X$ on the outer face. Follow Figure 8.

(Necessity of condition (i)): If v is a source in F , then there exists some w_t in F and vertex disjoint paths $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ by Proposition 6.3; so $E' = E \cup M_t$ is homeomorphic to a subgraph of G and is upward planar. If v is a sink in F , then u is the single source of F , as only u and v are possible sources. Thus, in F , there is a path $u \xrightarrow{\pm} v$, so $E' = E \cup M_{uv}$ is homeomorphic to a subgraph of G and is upward planar. If v is neither a source nor a sink in F then, by Lemma 6.6, there is also some $w_t > v$ and disjoint directed paths $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ in G . Since v is a non-source

in F , there is also a $u \xrightarrow{\pm} v$ path in F . This path crosses the $u \xrightarrow{\pm} w_t$ path at some latest vertex z on that path, so $E \cup (u \xrightarrow{\pm} z) \cup (z \xrightarrow{\pm} v) \cup (z \xrightarrow{\pm} w_t) \cup (v \xrightarrow{\pm} w_t)$ is a subgraph of G and hence upward planar. Note that these four paths are disjoint. Since z has in-degree one we can contract the $u \xrightarrow{\pm} z$ path to u without destroying upward planarity, by Lemma 5.3, so $E \cup \{(u, v), (u, w_t), (v, w_t)\}$ has an upward planar subdivision and is upward planar itself. By Lemma 6.2 F lies in a single face of Γ_E , so no other vertices lie inside the u, v, w_t triangle, and the extra edges and vertex for M_{uv} can be added without destroying planarity³. If some x_i is in F then, by the same argument as Theorem 6.5, all of F must be in the outer face of Γ_E . The marker, hence w_t or the (u, v) edge as appropriate, is therefore in the outer face of the drawing induced by E on Γ .

(Necessity of condition (ii)): If v is a source in E , then, by Proposition 6.3, there are vertex disjoint paths $s \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ in E . There must be an $s \xrightarrow{\pm} u$ path in E , otherwise there is either a second source (u is a source in F , so it cannot also be a source in E) or a cycle in G ($u < v$ in G , so there can be no $v \xrightarrow{\pm} u$ directed path in E). Let z be the last vertex common to paths $s \xrightarrow{\pm} u$ and $s \xrightarrow{\pm} w_t$. Then, $F \cup \{(z, u), (z, w_t), (v, w_t)\}$ is homeomorphic to a subgraph of G and is upward planar. Since $\deg^- u = 1$ (in this digraph), the edge (z, u) can be contracted without destroying upward planarity, by Lemma 5.3, and $F' = F \cup M_t$ is upward planar.

Otherwise (v a non-source), if $u < v$ in E , then $F' = F \cup M_{uv}$ is homeomorphic to a subgraph of G and, hence, is upward planar. If u and v are incomparable in E , then they share a greatest lower bound w_s , by Lemma 6.4, and $F \cup \{(w_s, u), (w_s, v)\}$ is upward planar. Again, $\deg^- u = 1$ in F , so the (w_s, u) edge can be contracted to give $F' = F \cup M_{uv}$.

The requirement for the E -marker to be on the outer face of Γ induced by F follows as before: Γ_E lies entirely within one face of Γ_F , and this is necessarily the outer face since E contains the source s .

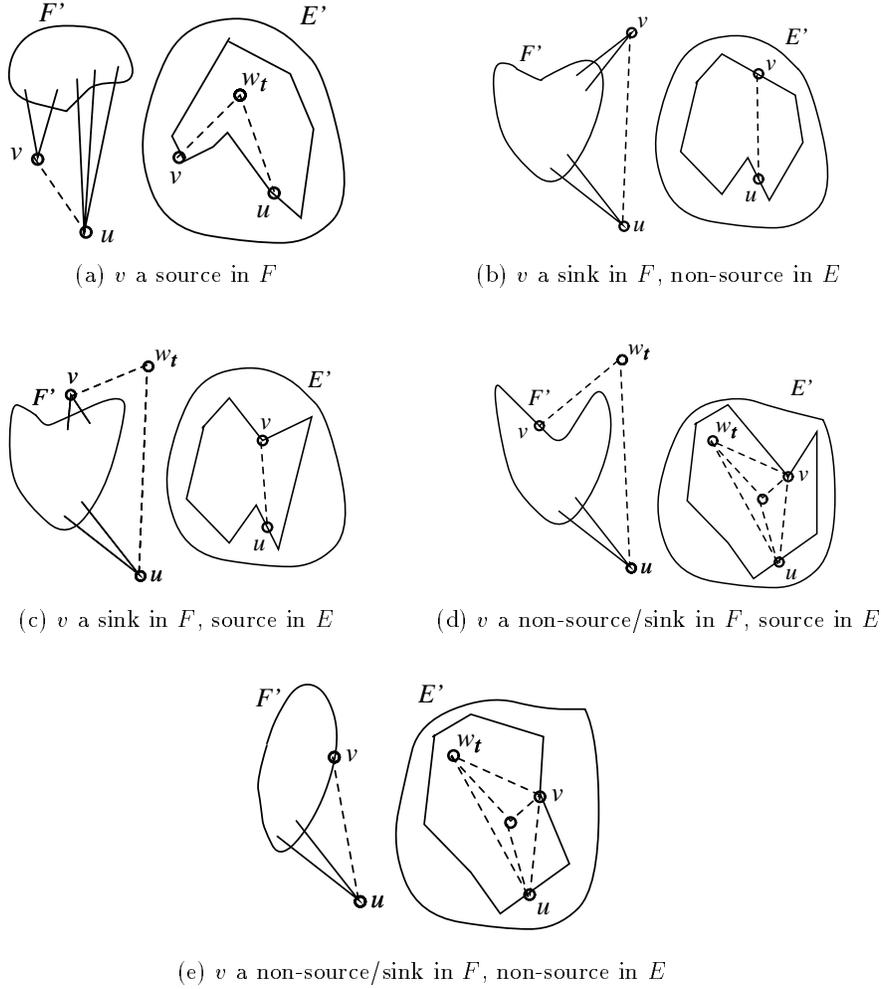
(Sufficiency) Suppose E' and F' admit upward plane drawings meeting the requirements (i) and (ii).

Case 1: v is a source in F : (See Figure 8(a).) If v is a source in F it cannot at the same time be a source in E , as $u < v$ in either E or F . Thus $F' = F \cup (u, v)$ is upward planar with single source u . Using Lemma 5.4, add F' (with u and v renamed as u' and v') to E' , identifying u' with w_t . We can do this so that edges (v, w_t) and (w_t, v') are consecutive in the cyclic order about w_t . Using Lemma 5.6, split w_t by making these two edges incident with a new vertex u_1 and the remaining edges incident with a new vertex u_2 . Now v' and u_1 have in-degree 1, so use Lemma 5.3 to contract their in-edges, thus identifying v and v' . Vertex u_2 has in-degree 1 so contract (u, u_2) . The result is the digraph G , and thus G is upward planar.

Case 2: v is a sink in F : (The two possibilities are illustrated in Figure 8 (b) and (c).) If v is a non-source in E , then $F' = F \cup (u, v)$ is upward planar with u and v on the outer face by assumption. If v is a source in E , then $F' = F \cup M_t$ is upward planar with w_t on the outer face. In either case F is upward planar with single source u and sink v on the outer face. By Lemma 5.5 we can add F to E' in place of the (u, v) edge in E' , and the result, G , is upward planar.

Case 3: v is a non-source/sink in F : (See Figure 8(d).) Suppose v is a source in E . Then $F' = F \cup M_t$ is upward planar with the sink w_t on the outer face. Using Lemma 5.5, add F' (renaming u and v to u' and v' respectively) to E' in place of the

³ The point of adding these edges is to fix the face in E for the sufficiency conditions.

FIG. 8. Merging E and F ; cut-set $\{u, v\}$ and $u < v$.

edge (u, v) , identifying u' with u and w_t with v . Throw away the edge (u, w_t) and the remaining marker edges of E' . Vertex v now has in-degree 1 so the edge (v', v) can be contracted by Lemma 5.3, and the result, G , is upward planar. Note that the M_{uvt} marker attached to E' is stronger than we actually require here (M_{uv} would do), but it necessarily does exist (as previously proven) and is needed for the next part of this case.

Suppose then that v is a non-source in E . Consider the upward planar representation of E' and throw away the marker edges, save for (u, w_t) , (v, w_t) , (u, v) , which then form a face. $F' = F \cup (u, v)$ is upward planar with u and v on the outer face. Let z be some sink on the outer face, and add the edge (v, z) to obtain F'' , upward planar with u, v, z on the outer face. Using Lemma 5.5, add F'' (with u and v renamed to u' and v') to E' in place of the edge (u, w_t) , identifying u' with u and z with w_t . Do this so that v' and v share the face of edges (u, v') , (v', z) , (u, v) , (v, z) . Clearly we can

now identify the vertices v and v' . We obtain an upward planar digraph containing G as a subgraph. See Figure 8(e).

As in the proof of Theorem 6.5 we notice that all vertices on the outer face of E' are necessarily on the outer face of the combined drawing, and if some x_i exists in F , then it is on the outer face of F' , and is forced to the outer face of the combined drawing by the second part of condition (i). \square

6.3. Cut-set $\{s, v\}$. As mentioned in the introduction to Section 6 (see also Lemma 6.2), it is important to be able to distinguish the “inner” and “outer” components. The inner component will be embedded in a face of the outer one, and thus the inner component will have to have the marker on its outer face since this marker is a proxy for the outer component. If we have to check each component as a potential inner component, we must recursively solve two subproblems for each component, and an exponential time blowup results.

Until now, the outer component has been uniquely identified as the *source component*, since that component cannot lie within an internal face of any other component. If we have a cut-set of the form $\{s, v\}$ where s is the source, then we lose this restriction, so we handle it instead by requiring one of the components, E , to be 3-connected so that deciding if it can be the inner face does not require recursive calls. To decide if E can be the inner face we need to test if it satisfies the role of E in the previous theorem—i.e. has an upward planar representation with the marker on its outer face. This can be done in linear time using the algorithm of Section 4. If G has only cut-sets of the form $\{s, v\}$, then, for at least one such cut-set, one of the components will be triconnected. Given the list of cut-sets we can find such a cut-set and such a component in linear time using depth-first search.

We capture these ideas in terms of two theorems. One is applicable if the triconnected component, E , can be the inner component, and one if it cannot. E “can be” the inner component if and only if it satisfies the same conditions that the inner component F satisfied in the previous Theorem 6.7. The similarity of both of these theorems to Theorem 6.7 should be clear. Note that in the statement of these theorems, we continue to use u (redundant since $u = s$) for consistency with previous usage.

THEOREM 6.8. *Let G be a biconnected DAG with a single source s , and let $X = \{x_i\} \subseteq V(G)$ be a set of vertices. Let $\{u, v\}$ be a separation pair of G where $u = s$, E be a 3-connected component of G with respect to $\{u, v\}$, and F be the union of all other components of G with respect to $\{u, v\}$. If*

$E' = (E \cup F$ -marker) admits an upward plane drawing with w_t (if it exists, otherwise the edge (u, v)) and all vertices of X in E on the outer face,

then G admits an upward plane drawing with all vertices of X on the outer face if and only if

- (i) *$F' = (F \cup E$ -marker) admits an upward plane drawing with all vertices of X in F on the outer face, and w_t (if it exists, otherwise the edge (u, v)) also on the outer face if some $x \in X$ contained in E .*

where

$$E\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } E \\ M_{uv} & \text{if } v \text{ is a sink in } E \\ M_{vt} & \text{otherwise.} \end{cases}$$

and

$$F\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } F \\ M_{uv} & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the necessity of Theorem 6.7 with some (simplifying) modifications. We will be more brief except where differences exist. Note that F corresponds to S and E corresponds to H of the previous notation. The new notation is to emphasize that neither is the unique source-component.

(Necessity): Suppose G has an upward planar representation Γ with all x_i 's on the outer face, and that some upward planar representation $\Gamma_{E'}$ satisfies the precondition for E' . We need to show that the condition (i) holds:

If v is a source in E then there exist vertex disjoint paths $u \xrightarrow{\pm} w_t$ and $v \xrightarrow{\pm} w_t$ in E by Proposition 6.3, so $F' = F \cup M_t$ is homeomorphic to a subgraph of G and hence upward planar. If v is a sink in E , then u is the only source so $u \xrightarrow{\pm} v$ exists in E and $F' = F \cup M_{uv}$ is upward planar. Otherwise, v is neither and there exist vertex disjoint $u \xrightarrow{\pm} v$, $v \xrightarrow{\pm} w_t$ and $u \xrightarrow{\pm} w_t$ paths in E by Lemma 6.6 and the fact that $u = s$ is the single source in E , so $F' = F \cup \{(u, v), (u, w_t), (v, w_t)\}$ is upward planar. Since these three edges are a face in the representation of this digraph, the required extra edges and vertices can be added and the result is upward planar. The necessity of the x_i conditions follows as before, as, if some vertex of E is required on the outer face, then all of E is.

(Sufficiency): The sufficiency proof is exactly that of Theorem 6.7 (with $F \equiv S$ and $E \equiv H$). This is as the various constructions are not dependent on u differing from s . \square

THEOREM 6.9. *Let G be a biconnected DAG with a single source s and let $X = \{x_i\} \subseteq V(G)$ be a set of vertices. Let $\{u, v\}$ be a separation pair of G where $u = s$, E be a 3-connected component of G with respect to $\{u, v\}$, and F be the union of all other components of G with respect to $\{u, v\}$. If it is not true that*

$E^ = (E \cup F^*\text{-marker})$ admits an upward plane drawing with w_t (if it exists, otherwise the edge (u, v)) and all vertices of X in E on the outer face,*

where

$$F^*\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } F \\ M_{uv} & \text{otherwise.} \end{cases}$$

then G admits an upward plane drawing with all vertices of X on the outer face if and only if

- (i) *There is no $x \in X$ contained in F .*
- (ii) *$F' = (F \cup E\text{-marker})$ admits an upward plane drawing with w_t (if it exists, otherwise the edge (u, v)) on the outer face.*
- (iii) *$E' = (E \cup F\text{-marker})$ admits an upward plane drawing with all $x \in X$ on the outer face,*

where

$$E\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } E \\ M_{uv} & \text{otherwise.} \end{cases}$$

and

$$F\text{-marker} = \begin{cases} M_t & \text{if } v \text{ is a source in } F \\ M_{uv} & \text{if } v \text{ is a sink in } F \\ M_{uvt} & \text{otherwise.} \end{cases}$$

Proof. (outline) Since E has no upward plane drawing with u and v both appropriately on the outer face, by Lemma 6.2 the only way G could be upward planar is if F can be embedded within an internal (hence the new condition (i)) face of E . Thus, the outer face of G is fixed as being some face of the drawing of E' not containing v . It remains to ensure that there is some embedding of F which will fit the structural constraints of the shape of a face shared by s and v in the drawing of E . These are exactly the conditions previously required by E for embedding within the drawing of F . The remainder of the proof does not rely on the triconnectedness of either component, and is similar to the proof of Theorem 6.8 with the roles of E and F reversed. \square

6.4. The Algorithm. Here we briefly summarize our algorithm and discuss its complexity.

Given DAG G , we first isolate biconnected components using the algorithm discussed in [23]. By Lemma 6.1 these can be tested independently. This step requires $\Theta(n)$ time, plus the time to test each piece independently.

For a biconnected DAG G , we use the triconnected components algorithm of [13] to find a list of separation pairs which breaks G into triconnected components ($\Theta(n)$ time). This list can be rearranged in linear time so that separation pairs involving the source appear last.

For each cut-set $\{u, v\}$ where $u \neq s$, isolate the source component using depth-first search ($\Theta(n)$ time), apply the appropriate theorem (6.5 or 6.7) to add markers and partition X , then recurse. The required time is given by

$$T(n) = T(k) + T(n - k) + O(n) \quad (k \geq 1),$$

which is $O(n^2)$, assuming we can do the base-case on the smaller triconnected digraphs in $O(n^2)$ time.

Given a biconnected G with cut-set $\{s, v\}$, isolate a triconnected component (again with DFS) as E , and test, using the Algorithm of Section 4, to see if has an upward plane embedding as specified in Theorem 6.8; this requires only $O(n)$ time, since a triconnected digraph has at most 2 embeddings with two specified vertices on the outer face. If E passes, continue with testing F as per Theorem 6.8. Otherwise the operation is absorbed into the $O(n)$ term in the recurrence above, and we apply Theorem 6.9, forcing F to have u and v on the outer face. The previous recurrence and bound apply to this step.

Given a triconnected G , simply apply the algorithm of Section 4 to all possible outer faces, and test for upward planarity in $O(n^2)$ time.

If we wish to output our G embedded into a planar s - t digraph (for drawing), we must find the necessary set of augmenting edges. This can be done by simply running the final upward planar representation through the algorithm of Section 4.

The entire algorithm operates in $O(n^2)$ time, and correctness follows from the cited results. We have

THEOREM 6.10. *Testing a single-source acyclic digraph for upward planarity, and outputting an upward planar representation when it exists, can be done in $O(n^2)$ time.*

7. Conclusions and Further Work. We have given a linear time algorithm to test whether a given single-source digraph has an upward plane drawing strongly equivalent to a given plane drawing, and give a representation for this drawing if it

exists. This provides, in combination with the decomposition results of Sections 5 and 6, an efficient $O(n^2)$ algorithm to test upward planarity of an arbitrary single-source digraph. We have also given a combinatorial characterization of single-source upward planar digraphs which provides new insights into their structure.

A lower bound for the single-source upward planarity problem is not known, although we believe that it may be possible to perform the entire test in sub-quadratic (perhaps linear) time. An obvious extension of this work would be to find such an algorithm or prove a lower bound.

This paper has concentrated on the issues of efficiently testing for an upward plane drawing. However, with the planar representation which results, we can augment the digraph to a planar s - t digraph with our algorithm of Section 4, then apply the algorithm of Di Battista and Tamassia to generate an actual drawing in $O(n)$ arithmetic steps, or $O(n \log n)$ arithmetic steps for a straight-line drawing [7]. Since an actual drawing must specify physical coordinates for the vertices, it becomes relevant to ask how big the integer grid must be or, equivalently, how much real precision is required. If bends are allowed an $O(n)$ by $O(n)$ grid suffices [7], similar to the case of undirected planar graphs [4, 20]. There is no upper bound known on the worst case size requirement for straight-line upward plane drawings, but Di Battista, Tamassia and Tollis [8] have exhibited a class of upward planar digraphs requiring an $\Omega(2^n)$ sized integer grid. Thus, any straight-line drawing algorithm is output sensitive—individual coordinates could require $\Omega(n)$ bits causing the output size to dominate the arithmetic time. Without an upper bound on the required area, it is not known if the drawing algorithm remains polynomial time—any digraph requiring doubly-exponential coordinate size would then need exponential time. It would be interesting to characterize some classes of digraphs which permit straight-line upward plane drawings on a polynomially sized grid. Guaranteeing minimum area in all cases is, however, NP-hard [9].

The more general problem, testing upward planarity of an arbitrary acyclic digraph, has recently been shown to be NP-complete [12]. Another recent development by Bertolazzi and Di Battista [1] shows how to efficiently test a triconnected (multi-source, multi-sink) DAG for upward planarity, a more general analogue of the result in Section 4.

REFERENCES

- [1] P. BERTOLAZZI AND G. DI BATTISTA, *On upward drawing testing of triconnected digraphs*, in Proc. 7th ACM Symposium on Computational Geometry, 1991, pp. 272–280. Also TR RAP.18.90, Dipartimento di Informatica e Sistemistica, Università degli Studi di Roma, “La Sapienza”.
- [2] J. A. BONDY AND U. S. R. MURTY, *Graph Theory with Applications*, MacMillan Co., New York, NY, 1976.
- [3] K. S. BOOTH AND G. S. LUEKER, *Testing the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms*, J. Comput. Syst. Sci., 13 (1976), pp. 335–379.
- [4] H. DE FRAYSSEIX, J. PACH, AND R. POLLACK, *Small sets supporting Fáry embeddings of planar graphs*, in Proc. 20th ACM Symposium on the Theory of Computing (STOC), 1988, pp. 426–433.
- [5] G. DI BATTISTA, P. EADES, R. TAMASSIA, AND I. G. TOLLIS, *Algorithms for drawing graphs: an annotated bibliography*, November, 1993. Preprint, Dept. Comput. Sci., Brown Univ., Providence, RI. Preliminary version available via anonymous ftp from `ftp.cs.brown.edu:pub/papers/compgeo/gdbiblio.tex.Z` and `gdbiblio.ps.Z`.
- [6] G. DI BATTISTA, W. LIU, AND I. RIVAL, *Bipartite graphs, upward drawings, and planarity*, Inf. Proc. Letters, 36 (1990), pp. 317–322.

- [7] G. DI BATTISTA AND R. TAMASSIA, *Algorithms for plane representations of acyclic digraphs*, Theoretical Computer Science, 61 (1988), pp. 175–178.
- [8] G. DI BATTISTA, R. TAMASSIA, AND I. G. TOLLIS, *Area requirement and symmetry display of planar upward drawings*, Discrete Comput. Geom., 7 (1992), pp. 381–401.
- [9] D. DOLEV, F. T. LEIGHTON, AND H. TRICKEY, *Planar embedding of planar graphs*, in Advances in Computing Research, vol. 2. F. P. Preparata (Ed.), JAI Press Inc., Greenwich, Connecticut, 1984, pp. 147–161.
- [10] I. FÁRY, *On straight line representations of planar graphs*, Acta. Sci. Math. Szeged, 11 (1948), pp. 229–233.
- [11] I. S. FILOTTI, G. L. MILLER, AND J. REIF, *On determining the genus of a graph in $O(v^{O(g)})$ steps*, in Proc. 11th ACM Symposium on the Theory of Computing (STOC), 1979, pp. 27–37.
- [12] A. GARG AND R. TAMASSIA, *On the computational complexity of upward and rectilinear planarity testing*, 1994. Report CS-94-10. Dept. Comput. Sci., Brown Univ., Providence, RI.
- [13] J. HOPCROFT AND R. E. TARJAN, *Dividing a graph into triconnected components*, SIAM J. Computing, 2 (1972), pp. 135–158.
- [14] ———, *Efficient planarity testing*, J. ACM, 21:4 (1974), pp. 549–568.
- [15] M. D. HUTTON, *Upward planar drawing of single source acyclic digraphs*, master's thesis, University of Waterloo, Waterloo, Ontario, Canada, Sept., 1990.
- [16] M. D. HUTTON AND A. LUBIW, *Upward planar drawing of single source acyclic digraphs*, in Proc. 2nd ACM/SIAM Symposium on Discrete Algorithms (SODA), 1991, pp. 203–211.
- [17] D. KELLY, *Fundamentals of planar ordered sets*, Discrete Math, 63 (1987), pp. 197–216.
- [18] D. KELLY AND I. RIVAL, *Planar lattices*, Can. J. Math, 27:3 (1975), pp. 636–665.
- [19] C. R. PLATT, *Planar lattices and planar graphs*, J. Comb. Theory (B), 21 (1976), pp. 30–39.
- [20] W. SCHNYDER, *Embedding planar graphs on the grid*, in Proc. 1st ACM/SIAM Symposium on Discrete Algorithms (SODA), 1990, pp. 138–148.
- [21] S. K. STEIN, *Convex maps*, Proc. Amer. Math. Soc., 2 (1951), pp. 464–466.
- [22] E. STEINITZ AND H. RADEMACHER, *Vorlesungen über die Theorie der Polyeder*, Julius Springer, Berlin, Germany., 1934.
- [23] R. E. TARJAN, *Depth-first search and linear graph algorithms*, SIAM J. Computing, 1:2 (1972), pp. 145–159.
- [24] C. THOMASSEN, *Planar acyclic oriented graphs*, Order, 5 (1989), pp. 349–361.
- [25] W. T. TUTTE, *Convex representations of graphs*, Proc. London Math. Soc., 10 (1960), pp. 304–320.
- [26] K. WAGNER, *Bemerkungen zum vierfarbenproblem*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 46 (1936), pp. 26–32.
- [27] W. T. TROTTER, ED., *Planar Graphs: DIMACS Series in Discrete Mathematics and Computer Science, Volume 9.*, American Mathematical Society, Providence, RI, 1993.