1 Review of the Sampling Theorem

We previously stated the sampling theorem, which allows us to reconstruct a continuous-time signal from its discrete-time samples under certain conditions.

The Sampling Theorem states that, for a continuous-time (CT) signal $x_c(t)$ with CT Fourier transform $X_c(\omega)$, if $X_c(\omega) = 0$ for $|\omega| > \omega_m$ and $\omega_s > 2\omega_m$ then

$$x_d(n) = x_c(nT_s) = x_c(t)|_{t=nT_s}, n \in \mathbb{Z}$$

completely represents $x_c(t)$ and $x_c(t)$ can be reconstructed from $x_d(n)$.

In other words, we need to sample fast enough to get more than 2 samples in each cycle of $x_c(t)$’s highest frequency. So, if $T_s < \frac{1}{2}T_m$, then $x_c(nT_s)$ represents $x_c(t)$.

2 The Sampling Theorem in Action

Let’s thing about applying the sampling theorem. Consider $x_c(t) = \cos \omega_m t$ which has fundamental period $T_m$.  

\[ x_c(t) = \cos(\omega_m t) \]
Sampling and the Sampling Theorem

Recall that the Fourier transform of $X_c(\omega) = \mathcal{F}\{x_c(t)\} = \mathcal{F}\{\cos(\omega m t)\}$ so

$$X_c(\omega) = \pi \delta(\omega - \omega_m) + \pi \delta(\omega + \omega_m).$$

Obviously in this case, $X_c(\omega) = 0$ for $|\omega| > \omega_m$, so the signal $x_c(t)$ is band-limited. Let’s choose a value for $\omega_s$ so that $\omega_s > 2\omega_m$ and therefore

$$\frac{2\pi}{T_s} > 2 \cdot \frac{2\pi}{T_m},$$

or in other words $T_m > 2T_s$ or $T_s < \frac{1}{2}T_m$ and then we can plot as below.

We can kind of see how the samples represent $x_c(t)$.

Now consider the plot below.
Sampling and the Sampling Theorem

Let’s assume that there are two sets of samples taken, those marked with ‘x’, and those marked with ‘o’. We will also assume that in each case, $\omega_s > 2\omega_m$ so our sampling rate is high enough.

Clearly, each set of samples will have different numerical values, since they record different points on the original CT signal. Yet the sampling theorem suggests that the CT signal can be reconstructed from either of the sets of samples.

How can this be? It seems counter-intuitive that different sets of samples will reproduce the same signal. But that is what happens because the samples are used to recover the lowest frequency signal. That basically means that there is only one possible signal that could have produced a given set of samples, so long as the assumptions of the sampling theorem were true.

3 Sampling Processes

We have seen, at least in a qualitative sense, how samples can represent a CT signal provided that we follow the rules of the sampling theorem. At this point, we should consider how we can get samples – after all, without being able to sample real-world CT signals, there’s not much point to this course. That is, we need to know how to get samples from a CT signal.
Sampling and the Sampling Theorem

There are many methods to get samples, some of which are covered in the following subsections.

3.1 Sample-and-Hold Circuit

The sample-and-hold circuit (sometimes called zero-order hold) converts the held analog values to digital values during the hold interval $T_s$, as shown in the following diagram.

![Diagram of Sample-and-Hold Circuit]

Essentially the circuit latches values of the analog signal during the hold intervals, producing a series of stepped samples.

3.2 Pulse Amplitude Modulation (PAM) Sampling

This type of sampling multiplies a CT signal $x_c(t)$ with a pulse signal $p(t)$ to get a series of pulses that map certain portions of the original CT signal.
Sampling and the Sampling Theorem

In the pulse train, note that the area of each pulse is 1 – that is, its height can be considered to be $\Delta$, and its width $1/\Delta$.

When the signal is modulated (multiplied) with the pulses, we get $x_p(t) = x_c(t) \cdot p(t)$ shown below.

Each pulse of the modulated signal has an area that is approximately equal to the value of the original signal at the middle of the pulse width, which is also the value of the DT sample we would ideally have. This is illustrated in the above diagram.
Sampling and the Sampling Theorem

Each modulated pulse above can be integrated to get its area, as follows.

\[ x_a(n) = \int_{nT_s-\delta}^{nT_s+\delta} x_p(t)dt \approx x_c(nT_s) \]

Thus, the area of each modulated pulse is roughly the value of the sample. So this is a good approximation, but is not exactly what we want.

But, what happens when the width of the pulse narrows, as \( \frac{1}{\Delta} \to 0 \)? Our pulse train \( p(t) \) becomes an impulse train, with each pulse being a Dirac delta, \( \delta(t) \). This suggests a theoretical approach to sampling, which we explore in the next section.

4 Ideal Impulse Train Sampling

Consider our pulse train \( p(t) \) from the previous section, with the width of the pulse going to 0 (and thus the height going to infinity).

As \( \frac{1}{\Delta} \to 0 \), pulse train \( \to \) impulse train.

So the impulse train is written as

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \]

which is a series of shifted deltas.

So, using this \( p(t) \), what is \( x_p(t) = x_c(t) \cdot p(t) \) now?

If we recall that

\[ \int_{-T_s/2}^{+T_s/2} p(t)dt = \int_{-T_s/2}^{+T_s/2} \delta(t)dt = 1 \]

then we get the following.

If our original CT signal is
Sampling and the Sampling Theorem

\[ x_d(n) = \int_{nT_s - \frac{T_s}{2}}^{nT_s + \frac{T_s}{2}} x_p(t) dt = x_c(nT_s) \]

a.k.a. train to samples. Note that the above is *exactly* equal, not approximately as with PAM sampling before.
Sampling and the Sampling Theorem

Therefore, this is our model of ideal sampling, which is the kind of sampling that the sampling theorem means.

\[ x_c(t) \xrightarrow{p(t)} x_p(t) \xrightarrow{\text{Convert impulse train to samples}} x_d(n) \]

This model above represents the basic concept of a C/D converter, illustrated below.

**Example 4.1: Basic Sampling Theorem Exercises**

Consider a CT signal \( x_c(t) \) with a Fourier transform \( X_c(\omega) \) that is sampled with ideal sampling, with \( T_s = 10^{-4} \) seconds. For each of the following scenarios, determine whether the sampling theorem guarantees that \( x_c(t) \) can be recovered from its samples.

1. What if \( X_c(\omega) = 0 \) for \( |\omega| > 5000\pi \)?

   Since \( T_s = 10^{-4} \) seconds, \( \omega_s = 2\pi/T_s = 2\pi/10^{-4} = 20000\pi \) or \( f_s = 1/T_s = 10000 \) Hz. In this scenario, \( \omega_m = 500\pi = 2\pi \cdot (2500) \).

   Is \( (\omega_s = 20000\pi) > (2\omega_m = 10000\pi) \)?

   \( \text{Yes!} \) Therefore, we can guarantee recovery of \( x_c(t) \).

2. What if \( X_c(\omega) = 0 \) for \( |\omega| > 10000\pi \)?

   In this case, \( \omega_m = 10000\pi \).

   Is \( (\omega_s = 20000\pi) > (2\omega_m = 20000\pi) \)?
Sampling and the Sampling Theorem

Obviously not. In this case, $\omega_s = 2\omega_m$ which does not satisfy the requirements of the sampling theorem, which requires strictly greater than, not greater than or equal. Therefore, we cannot guarantee recovery of $x_c(t)$.

3. What if $X_c(\omega) = 0$ for $|\omega| > 15000\pi$?

Here, $\omega_m = 15000\pi$.

Is $(\omega_s = 20000\pi) > (2\omega_m = 30000\pi)$?

Obviously not. Therefore, we cannot guarantee recovery of $x_c(t)$.

4. What if $|X_c(\omega)| = 0$ for $|\omega| > 5000\pi$?

This question isn’t quite so obvious. Can we figure out, from the given information, whether the signal is band-limited? And if so, can we figure out whether the sampling was fast enough?

Consider that since $X_c(\omega) = |X_c(\omega)| e^{j\Theta(\omega)}$, we can be sure that $X_c(\omega) = 0$ for $|\omega| > 5000\pi$ since in that range, $X_c(\omega) = 0 \cdot e^{j\Theta(\omega)} = 0$.

So signal is band-limited, and $\omega_m = 5000\pi$.

Is $(\omega_s = 20000\pi) > (2\omega_m = 10000\pi)$?

Yes. Therefore, we can guarantee recovery of $x_c(t)$.

5. What if $\Re\{X_c(\omega)\} = 0$ for $|\omega| > 5000\pi$?

Again, this is not an obvious question. Consider this representation of $X_c(\omega)$.

$$X_c(\omega) = \Re\{X_c(\omega)\} + j\Im\{X_c(\omega)\}$$

Here, we know that the real part is zero outside of the giving band limits but we know nothing about the imaginary part. Thus, we can’t say for sure whether it will be zero or non-zero.

Since we don’t know for sure whether $X_c(\omega) = 0$ for $|\omega| > 5000\pi$, we cannot guarantee recovery.

6. What if $X_c(\omega) \ast X_c(\omega) = 0$ for $|\omega| > 15000\pi$?

This is another tricky situation. What can we figure out about $X_c(\omega)$?

Thinking back to previous signals courses, recall that

$$X_c(\omega) \ast X_c(\omega) \downarrow \text{Fourier Transform}$$

$$2\pi x_c(t) x_c(t)$$

and we can illustrate what happens in this scenario.
Recall: 

In this case, the band limits of the convolved transforms of $X_c(\omega)$ are twice those of $X_c(\omega)$ itself.

So $X_c(\omega) = 0$ for $|\omega| > 7500\pi$, so $\omega_m = 7500\pi$.

Is $(\omega_s = 20000\pi) > (2\omega_m = 15000\pi)$?

Yes. Therefore, we can guarantee recovery of $x_c(t)$.

Additionally, for self-practice, you could identify the range of values of $\omega_s$ for each scenario that would allow the signal to be recovered.

5 Analysis of Ideal Sampling

In the preceding sections, we have shown our model of ideal sampling, in which the CT signal $x_c(t)$ is modulated by an impulse train $p(t)$ to produce $x_p(t)$. Naturally, we want to know whether $x_p(t)$ really represents $x_c(t)$ uniquely and completely.

Since the sampling theorem talks about band-limited signals, it makes sense to do some frequency analysis. First, we’ll do a quick review of the Fourier transform.

Aside: Review of Fourier Transform

Recall the Fourier transform $X_c(\omega)$ of CT signal $x_c(t)$ is given by:

$$X_c(\omega) = \mathcal{F}\{x_c(t)\} = \int_{-\infty}^{+\infty} x_c(t)e^{-j\omega t}dt, \omega \in \mathbb{R}$$

This has the inverse transform given by:

$$x_c(t) = \mathcal{F}^{-1}\{X_c(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_c(\omega)e^{j\omega t}d\omega$$
Sampling and the Sampling Theorem

The above is true provided that \( x_c(t) \) satisfies some conditions (such as being finite energy) or if \( x_c(t) \) is periodic with a non-standard Fourier transform.

So, for our frequency analysis, let us assume our sampling function is

\[
p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)
\]

and that \( x_c(t) \) is any CT signal that has a CT Fourier transform.

Consider \( x_p(t) \), which is still a CT signal as well.

\[
x_p(t) = x_c(t) \cdot p(t)
\]

\[
= x_c(t) \cdot \left[ \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \right], n \in \mathbb{Z}
\]

\[
= \left[ \sum_{n=-\infty}^{+\infty} x_c(t) \delta(t - nT_s) \right], n \in \mathbb{Z}
\]

\[
= \left[ \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s) \right], n \in \mathbb{Z}
\]

Note that the values of \( x_c(nT_s) \) are the samples.

Now we analyze in the frequency domain. Recall that,

\[
x(t) \cdot y(t) \quad \overset{\text{Fourier Transform}}{\downarrow} \quad \frac{1}{2\pi} X(\omega) * Y(\omega)
\]

so

\[
X_p(\omega) = \mathcal{F}\{x_c(t) \cdot p(t)\}
\]

\[
= \frac{1}{2\pi} X_c(\omega) * P(\omega)
\]

but that leads to the question, “What is \( P(\omega) \)?” That is, what is the Fourier transform of \( P(\omega) \)?

\[
P(\omega) = \mathcal{F}\left\{ \sum_n \delta(t - nT_s) \right\}
\]

We know that \( p(t) \) is periodic, with period \( T_s \), so \( P(\omega) \) is a non-standard transform based on its Fourier series.
Sampling and the Sampling Theorem

Aside: Fourier Series

For a periodic signal $x(t)$ with period $T_o$, there is a Fourier series with coefficients

$$a_k = \frac{1}{T_o} \int_{T_o} x(t)e^{jk\omega_o t} dt \quad \text{with} \quad \omega_o = \frac{2\pi}{T_o}$$

and therefore a non-standard Fourier transform, shown here.

$$X(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_o)$$

So, based on our knowledge of $p(t)$ and Fourier transforms and series,

$$P(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_s) \quad \text{where} \quad \omega_s = \frac{2\pi}{T_s}$$

where the values of $a_k$ are determined by

$$a_k = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} p(t)e^{-jk\omega_s t} dt$$

$$= \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)e^{-jk\omega_s t} dt$$

$$= \frac{1}{T_s} e^{-jk\omega_s t} \bigg|_{t=0}$$

$$= \frac{1}{T_s}$$

which gives us a final expression for $P(\omega)$, shown and illustrated below.

$$P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) \quad \text{with} \quad \omega_s = \frac{2\pi}{T_s}$$

\[\begin{array}{c}
\cdots -\omega_s \\
\omega_s \\
2\omega_s \\
3\omega_s \\
\cdots
\end{array}\]
Now that we know $P(\omega)$, we know

$$X_p(\omega) = \frac{1}{2\pi} X_c(\omega) * P(\omega)$$

$$= \frac{1}{2\pi} X_c(\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(\omega) * \delta(\omega - k\omega_s)$$

which gives

$$X_p(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s)$$

since $x(t) * \delta(t - t_o) = x(t - t_o)$.

So $X_p(\omega)$ is a summation of scaled and shifted versions of $X_c(\omega)$. This is an important result.

Now, let's consider an $x_c(t)$ and assume the sampling theorem conditions are met, so that $X_c(\omega) = 0$ for $|\omega| > \omega_m$ and $\omega_s > 2\omega_m$.

$$X_c(\omega) = \begin{cases} 
\text{anything} & |\omega| < \omega_m \\
0 & |\omega| > \omega_m 
\end{cases}$$

An example of a possible $X_c(\omega)$ is shown in the following figure.

With the $X_c(\omega)$ shown above, $X_p(\omega)$ would be as illustrated below, since it is made of shifted and scaled replicas of $X_c(\omega)$:

$$X_p(\omega) = \ldots + \frac{1}{T_s} X_c(\omega + 2\omega_s) + \frac{1}{T_s} X_c(\omega + \omega_s) + \frac{1}{T_s} X_c(\omega) + \frac{1}{T_s} X_c(\omega - \omega_s) + \frac{1}{T_s} X_c(\omega - 2\omega_s) + \ldots$$
Sampling and the Sampling Theorem

We have non-overlapping replicas of $X_c(\omega)$ so that we can recover $X_c(\omega)$ from $X_p(\omega)$ by defining a reconstruction filter, $H_r(\omega)$, defined and illustrated as follows.

\[
H_r(\omega) = \begin{cases} 
T_s & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\
0 & |\omega| > \frac{\omega_s}{2}
\end{cases}
\]

Then, if the recovered signal $X_r(\omega) = X_p(\omega) \cdot H_r(\omega)$ then $X_r(\omega) = X_c(\omega)$ and therefore $x_r(t) = x_c(t)$. Thus $x_c(t)$ is recovered from its samples!
Sampling and the Sampling Theorem

The reconstruction occurs because it ignores all but one of the replicas (the lowest frequency one) and undoes the scaling of \( X_c(\omega) \) that occurs in \( X_p(\omega) \) for that replica.

So this is how the reconstruction works when the sampling theorem conditions are met. What happens if the conditions aren’t met? We’ll do two examples to explore that scenario.

Example 5.1: \( x_c(t) \) is band-limited but undersampled

Suppose we have a CT signal \( x_c(t) \) with Fourier transform \( X_c(\omega) \), that is band-limited. However, suppose we sample at less than the Nyquist rate. That is, let \( \omega_s \neq 2\omega_m \).

Let \( X_c(\omega) \) be defined as in the figure below.

Also, recall the value of \( X_p(\omega) \).

\[
X_p(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s)
\]

With \( \omega_s \neq 2\omega_m \), \( X_p(\omega) \) would be as shown below. Note that the replicas overlap, and thus add, in places. The added version is shown by a dotted line where it differs from the non-adding parts.
Sampling and the Sampling Theorem

Let’s try to recover $X_c(\omega)$ using $H_r(\omega)$ by applying it to $X_p(\omega)$.

$$H_r(\omega) = \begin{cases} 
T_s & |\omega| < \frac{\omega_s}{2} \\
0 & |\omega| > \frac{\omega_s}{2} 
\end{cases}$$

So now $X_r(\omega) = H_r(\omega) \cdot X_p(\omega)$.

The above $X_r(\omega)$ is clearly not the same as $X_c(\omega)$, and so $x_r(t) \neq x_c(t)$ and the original signal is not recovered. This is called aliasing due to undersampling.

Also, as an item of note, in this case, if $\omega_s = 2\omega_m$ exactly, we still would have been able to recover $X_c(\omega)$. But that is only because, in this case, $X_c(\omega) = 0$ at the band-limits, and thus there is no interference. That’s why the sampling theorem can only guarantee recovery for $\omega_s > 2\omega_m$, since $\omega_s = 2\omega_m$ depends on the values of $X_c(\omega)$.

In the prior example, since the sampling rate was too low to meet the conditions of the sampling theorem, we had overlapping replicas which added together and thus obscured the original value. This is called aliasing.
Example 5.2: sin function

Let \( x_c(t) = \sin \omega_o t \) and thus

\[
X_c(\omega) = \frac{\pi}{j} \delta(\omega - \omega_o) - \frac{\pi}{j} \delta(\omega + \omega_o)
\]

It is clear that \( X_c(\omega) \) is band-limited to \( \omega_m = \omega_o \).

Let us consider the possible scenarios for the sampling theorem.

1. When \( \omega_s > 2\omega_m \),

We have non-overlapping replicas, and a simple reconstruction filter \( H_r(\omega) \) as seen previously can reconstruct. This is not surprising, since we have obeyed the rules of the sampling theorem.
2. When $\omega_s < 2\omega_m$, 

Here, our replicas overlap. When we apply the reconstruction filter, we do not recover the same signal. In fact, in this case, $x_r(t) = \sin(\omega_s - \omega_o)t = -\sin(\omega_s - \omega_o)t$ which is quite different from $x_c(t) = \sin \omega_o t$. Under this sampling scheme, $x_r(t)$ is an alias of $x_c(t)$.

3. When $\omega_s = 2\omega_m$, 

In this case, even if weird stuff didn’t happen due to the undefined boundary of the reconstruction filter, the replicas clearly cancel each other out. Thus, the recovered signal will have no frequency components. $x_r(t)$ will be a DC signal, which is clearly wrong.

We see aliasing again in this example. Aliasing can be avoided by following the requirements of the sampling theorem.

Signals and Systems, 2E: Chapter 7.0 to 7.3, page(s) 514–534.
Signal Processing First: Chapter 4-1, 4-5, page(s) 71–79, 93–94.

We’ll conclude with an example covering ideal sampling, before looking at the concept of aliasing in a bit more detail.
Example 5.3: Ideal Sampling

Given that

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{and} \quad T_s = \frac{1}{6000}, \]

for each of the following signals, plot \( X_p(\omega) \) which is the Fourier transform of \( x_p(t) = x_c(t) \cdot p(t) \), then determine what happens when that signal is reconstructed by the ideal reconstruction filter \( H_r(\omega) \) with cut-off frequency \( \omega_s/2 \), as defined below.

\[
H_r(\omega) = \begin{cases} 
T_s & |\omega| < \frac{\omega_s}{2} \\
0 & |\omega| > \frac{\omega_s}{2} 
\end{cases}
\]

1. Case 1: \( x_c(t) = \frac{\sin 4000\pi t}{\pi t} \)

   Here, we can use the CT Fourier Transform tables in the textbooks to figure out what \( X_c(\omega) \) will be.

\[
x_c(t) = \frac{\sin 4000\pi t}{\pi t} \quad \downarrow \quad \text{Fourier Transform} \\
X_c(\omega) = \begin{cases} 
1 & |\omega| < 4000\pi \\
0 & |\omega| > 4000\pi 
\end{cases}
\]

This is clearly band-limited, with \( \omega_m = 4000\pi = 2\pi \cdot 2000 \) and \( T_m = 1/2000 \). Since \( (\omega_s = 2\pi/T_s = 12000\pi) > (2\omega_m = 8000\pi) \), this meets the requirements of the sampling theorem, and so no aliasing should occur. Therefore, \( X_p(\omega) \) is simply a set of shifted, scaled replicas of \( X_c(\omega) \), as shown below.

\[
X_p(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s)
\]

This means that the ideal reconstruction filter will successfully recover the original signal. In other words, \( X_r(\omega) = X_p(\omega) \cdot H_r(\omega) = X_c(\omega) \).
2. Case 2: \( x_c(t) = 1 + \cos 2000\pi t + \sin 4000\pi t \)

Here, we can use the CT Fourier Transform tables in the textbooks to figure out what \( X_c(\omega) \) will be.

\[
x_c(t) = 1 + \cos 2000\pi t + \sin 4000\pi t \\
\xrightarrow{\text{Fourier Transform}} \\
X_c(\omega) = 2\pi \delta(\omega) + \pi \delta(\omega - 2000\pi) + \pi \delta(\omega + 2000\pi) \\
+ \frac{\pi}{j} \delta(\omega - 4000\pi) - \frac{\pi}{j} \delta(\omega + 4000\pi)
\]

This is also clearly band-limited, with \( \omega_m = 4000\pi = 2\pi \cdot 2000 \) and \( T_m = 1/2000 \). Since \( (\omega_s = 2\pi/T_s = 12000\pi) > (2\omega_m = 8000\pi) \), this meets the requirements of the sampling theorem, and so no aliasing should occur. Therefore, \( X_p(\omega) \) is simply a set of shifted, scaled replicas of \( X_c(\omega) \), as shown below.

\[
X_p(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s)
\]

This means that the ideal reconstruction filter will successfully recover the original signal. In other words, \( X_r(\omega) = X_p(\omega) \cdot H_r(\omega) = X_c(\omega) \).

More examples involving aliasing will follow in the next section.