Discrete-Time Signals

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1 Discrete-Time Signals and Systems

A typical DT system might be represented as follows.

\[ x_d(n) \rightarrow \text{DT} \rightarrow y_d(n) \]

In this scenario, there are several points of interest.

- \( x_d(n) \) may be sampled from CT; furthermore, “special” signals exist.
- \( y_d(n) \) might be intended for reconstruction to CT.
- The discrete-time system is usually LTI, and thus represented by its impulse response.

We can analyze such systems with:

1. convolution;
2. the Discrete-Time Fourier Transform (DTFT), DT Fourier Series (DTFS), frequency response;
3. the z-Transform or system function.

Let us first consider DT signals before worrying about how DT systems work.

2 Introduction to DT Signals

DT signals can be expressed in several ways: functionally, piece-wise, or even just as a collection of values. This makes them much easier to deal with – in CT, we could never possibly express all values of even an finite-length signal.
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Consider the following signal $x(n)$, where $n \in \mathbb{Z}$.

$$x(n) = \begin{cases} 
-1 & \text{for } n = -2 \\
2 & \text{for } n = -1 \\
5 & \text{for } n = 0 \\
3 & \text{for } n = 1 \\
0 & \text{else}
\end{cases}$$

This is shown in the following figure.

This particular signal $x(n)$ is defined over all values of $n$, but only has a few that are non-zero, and thus can be represented by just a few values. It would be impossible to represent a CT signal like this.

Consider an alternate $x(n)$, shown below.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) = \begin{cases} 
\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \\
0 & \text{for } n < 0
\end{cases}$$

This is shown in the following figure.
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This particular signal \( x(n) \) is defined as a function. There are many different ways to represent DT signals.

2.1 Properties of DT Signals

Let us consider several properties of DT signals.

2.1.1 Energy of a Signal

We have two ways to consider the energy of a signal.

The time-limited energy refers to the signal’s energy over a finite period of time, and is expressed as follows.

\[
E_{N_1,N_2} = \sum_{n=N_1}^{N_2} |x(n)|^2
\]

The total energy refers to the signal’s energy over all time. We express total energy as follows.

\[
E_{\infty} = \sum_{n=-\infty}^{\infty} |x(n)|^2
\]

Aside:

This concept corresponds to signal energy in CT:

\[
E_{\infty} = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt
\]

2.1.2 Power of a Signal

Similarly to energy, when considering the power of a signal, we approach in one of two ways.

The time average power of a signal is given as follows.

\[
P_{N_1,N_2} = \frac{E_{N_1,N_2}}{N_2 - N_1 + 1}
\]

The total power of a signal is given by the following expression.

\[
P_{\infty} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2
\]
Aside: Signal Types

For a particular signal, if \( E_\infty < \infty \), we call that signal an energy signal.

If \( P_\infty < \infty \) but \( E_\infty = \infty \), we call the signal a power signal.

2.1.3 Periodicity of a Signal

A signal \( x(n) \) is periodic if there exists some \( N \in \mathbb{Z} \) such that \( x(n) = x(n + N) \) for \( \forall n \in \mathbb{Z} \).

Note that there may be many possible values of \( N \) that meet the above criteria. (For example, any multiple of a valid \( N \) will also be valid). The smallest such positive integer is called the fundamental period, and is represented as \( N_o \).

3 Basic DT Signals

As in CT, there are a number of basic signals that we define that, in turn, can be used to define more complex signals. Often, difficulties present in such signals in CT are avoided in DT.

3.1 Unit Impulse

This is also sometimes called the unit sample. Is is a basic signal \( \delta(n) \) that has a value of 1 at time 0, and is otherwise 0. In DT, this is a well-defined signal (unlike the CT impulse, which has an infinite height), and thus is quite usable.

\[
\delta(n) = \begin{cases} 
1 & \text{for } n = 0 \\
0 & \text{for } n \neq 0 
\end{cases}
\]
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3.2 Unit Step

The unit step is a basic signal, $u(n)$, which has a value of 1 for all times greater than or equal to 0, and 0 for all times less than 0. As with the unit impulse, this is well defined in DT (as opposed to CT) since the value of the signal at $n = 0$ is known.

$$u(n) = \begin{cases} 
1 & \text{for } n \geq 0 \\
0 & \text{for } n < 0 
\end{cases}$$

Let’s think about $u(n)$ in more detail. Note that $u(n-1)$ is as shown below.

So we see that $u(n) - u(n-1) = \delta(n)$.

Aside: Comparison to CT

Note that this is much simpler than the equivalent relation in CT.

$$\delta(t) = \frac{d}{dt}u(t) = \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} \approx u(t + 1) - u(t) \quad \text{if} \quad \Delta t = 1$$
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Since we can express $\delta(n)$ as a difference of unit steps, we can also express the unit step as a sum of deltas.

\[
    u(n) = \sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 
        1 & \text{for } n \geq 0 \\
        0 & \text{for } n < 0
    \end{cases}
\]

If we recall the relation in CT,

\[
    u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \delta(t) * u(t) = \int_{0}^{\infty} \delta(t - \tau)d\tau
\]

we can see that in DT, the unit step can also be expressed as follows.

\[
    u(n) = \delta(n) + \delta(n - 1) + \delta(n - 2) + ... = \sum_{k=0}^{\infty} \delta(n - k)
\]

So, just as in CT, $\delta(n)$ and $u(n)$ are related. For example, $u(n) = \sum_{k=-\infty}^{n} \delta(k)$ or $u(n) = \sum_{k=0}^{\infty} \delta(n - k)$. 

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We could also write the following.

\[
    u(n) = \sum_{k=-\infty}^{-1} 0 \cdot \delta(n - k) + \sum_{k=0}^{\infty} 1 \cdot \delta(n - k)
    = \sum_{k=-\infty}^{\infty} u(k)\delta(n - k)
\]

This is the convolution sum of \(u(n)\) and \(\delta(n)\).

**Aside: Note similarity to CT**

Note the similarity to the CT convolution integral.

\[
    u(t) = u(t) \ast \delta(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)d\tau
\]

In CT, we use convolution integrals. In DT, we use convolution sums.

This concept of the convolution sum extends to any finite-values DT signal, \(x(n)\).

\[
    x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n - k) = x(n) \ast \delta(n)
\]

This means that \(x(n)\) can be expressed as a sum of shifted and scaled deltas.

\[
    x(n) = ... + x(-2)\delta(n + 2) + x(-1)\delta(n + 1) + x(0)\delta(n) + x(1)\delta(n - 1) + x(2)\delta(n - 2) + ... 
\]

So any DT signal can be represented as a sum of deltas. Thus, the unit impulse is obviously as useful in DT as in CT – probably moreso.
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In DT, $\delta(n)$ is a well-defined signal/function, but it still has the expected sifting properties.

\[ x(n)\delta(n) = x(0)\delta(n) \]

\[ x(n)\delta(n-n_o) = x(n_o)\delta(n-n_o) \quad \text{where} \quad n_o \in \mathbb{Z} \]

\[ x(n-n_o)\delta(n) = x(-n_o)\delta(n) \]

\[ \sum_{n=-\infty}^{\infty} \delta(n) = 1 \]

\[ \sum_{n=-\infty}^{\infty} x(n)\delta(n) = x(0) \]

\[ \sum_{n=-\infty}^{\infty} x(n)\delta(n-n_o) = x(n_o) \quad \text{where} \quad n_o \in \mathbb{Z} \]

The fact that $\delta(n)$ is so well-defined and versatile makes handling DT signals much easier than in CT. Hence our interest in DT signals and systems.

We’ll come back to convolution sums after looking at some more basic signals.

### 3.3 Rectangular Pulse

The rectangular pulse $r_N(n)$ has a value of 1 over a range of $N$ points, and is 0 everywhere else. This is shown below.

\[ r_N(n) = u(n) - u(n-N) = \begin{cases} 1 & \text{for } 0 \leq n \leq N \\ 0 & \text{else} \end{cases} \]

This is a straightforward extension of the two previous basic signals, a simple rectangular pulse of $N$ samples. Note that the pulse starts and stops at known points – the boundaries are defined here in DT, whereas in the CT counterpart to this signal, they are not.
3.4 Ramp

This particular signal will probably never come up, and thus it doesn’t really count as a basic signal in DT. But some of you may remember ramp signal in CT, so here is the obvious analogous signal in DT.

\[ x_{\text{ramp}}(n) = \begin{cases} 
  n & \text{for } n \geq 0 \\
  0 & \text{else}
\end{cases} \]

3.5 DT Sinusoids and Exponentials

The most complex class of signals that we consider “basic” are the DT sinusoids and exponentials.

Consider the **general DT exponential**:

\[ x(n) = A \alpha^n \quad \text{where scalars} \quad A, \alpha \in \mathbb{C} \]

Some of examples of DT exponentials are

\[ x(n) = 5 \left( \frac{1}{2} \right)^n \]

or

\[ x(n) = e^{-j5} \left( e^{j\frac{\pi}{2}} \right)^n \quad \text{where} \quad A = e^{-j5} \quad \text{and} \quad \alpha = e^{j\frac{\pi}{2}}. \]

**Aside: General CT Exponential**

Recall, in CT, the general exponential form is \( x(t) = Ae^{\alpha t} \).
Let’s consider some special cases, where $A$ and $\alpha$ are real. First we’ll look at cases where $\alpha \geq 0$.

- If $\alpha > 1$, then as $n \to \infty$, $x(n)$ is a rising exponential.

- If $0 < \alpha < 1$, then as $n \to \infty$, $x(n)$ is a fading exponential.

- If $\alpha = 0$ or $\alpha = 1$ exactly, $x(n)$ is a constant signal.
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Now, consider cases where $\alpha < 0$.

- If $\alpha < -1$, then as $n \to \infty$, $x(n)$ is a rising and oscillating exponential.

- If $-1 < \alpha < 0$, then as $n \to \infty$, $x(n)$ is a fading and oscillating exponential.

- If $\alpha = -1$ exactly, $x(n)$ is a constantly oscillating signal.
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For the above special cases, we were dealing with real values. Now consider another special case.

If $\alpha \in \mathbb{C}$ and $|\alpha| = 1$, then we can do the following.

$$x(n) = \alpha^n = (e^{j\omega_0})^n \text{ where } \omega_0 \in \mathbb{R}$$

$$= e^{j\omega_0 n}$$

$$= \cos \omega_0 n + j \sin \omega_0 n$$

Thus, we see that DT exponentials link to DT sinusoids. The general form of this relationship is as follows. Note that $A, \alpha \in \mathbb{C}$, $\omega_0$ is measured in radians/second, and $\phi$ is measured in radians.

$$x(n) = A\alpha^n = |A| e^{j\phi} (r e^{j\omega_0})^n$$

$$= |A| r^n e^{j(\omega_0 n + \phi)}$$

$$= |A| r^n \cos (\omega_0 n + \phi) + j |A| r^n \sin (\omega_0 n + \phi)$$

Let’s look at the properties of $e^{j\omega_0 n}$, while keeping in mind that they translate into properties of sinusoids via Euler’s relation.

3.5.1 Property 1: Periodicity in Time (n)

Recall that in CT, $x(t) = e^{j\omega_0 t}$ has a fundamental period $T_0 = \frac{2\pi}{|\omega_0|}$. However, in DT, $e^{j\omega_0 n}$ is not necessarily periodic. This is because in DT, the exponential can only be evaluated at discrete points.

If $x(n) = e^{j\omega_0 n}$ is periodic, then we know the following.

$$x(n) = x(n + N) \text{ for } N \in \mathbb{Z}^+, n \in \mathbb{Z}$$

$$e^{j\omega_0 n} = e^{j\omega_0 (n + N)}$$

$$e^{j\omega_0 n} = e^{j\omega_0 n} e^{j\omega_0 N}$$

The last statement is only true if $e^{j\omega_0 N} = 1$, or in other words if $\cos \omega_0 N = 1$ and $\sin \omega_0 N = 0$. This is only the case when $\omega_0 N$ is a multiple of $2\pi$.

$$\omega_0 n = 2\pi k \text{ for } k \in \mathbb{Z}$$

$$\frac{\omega_0}{2\pi} = \frac{k}{N}, \text{ a rational number}$$
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So \( x(n) = e^{j\omega_0 n} \) is periodic iff (if and only if) \( \frac{\omega_o}{2\pi} = \frac{k}{N} \) is a rational number, which gives a period of \( N \) in samples.

(Note that a rational number is a number that is a ratio of two integers. Any such number is rational, even those that can’t be expressed to an exact value. Irrational numbers are just those that can’t be expressed as a ratio of two integers, like \( \pi \) and \( e \)).

If we reduce \( \frac{k}{N} \) to \( \frac{\omega_o}{N_o} \), then \( N_o \) is called the fundamental period. We call \( 2\pi/N_o \) the fundamental frequency, and when \( k = 1 \), \( \omega_o = \frac{2\pi}{N_o} \).

It is important to note that \( 2\pi/N_o \) is the rate of repetition of values, not the rate of oscillation (unlike in CT).

Let’s look at an example of this.

**Example 3.1:**

Consider the following signals.

\[ x_1(n) = \sin \left( \frac{\pi}{2} n \right) \]

For \( x_1(n) \), we see \( \omega_o = \pi/2 \), so \( \frac{\omega_o}{2\pi} = \frac{1}{4} \) and thus \( N_o = 4 \).

\[ x_2(n) = \sin \left( \frac{3\pi}{4} n \right) \]

For \( x_2(n) \), we see \( \omega_o = \frac{3\pi}{4} \), so \( \frac{\omega_o}{2\pi} = \frac{3}{8} \) and thus \( N_o = 8 \).

\[ x_3(n) = \sin \left( \frac{n}{6} \right) \]

For \( x_3(n) \), we see \( \omega_o = 1/6 \), so \( \frac{\omega_o}{2\pi} = \frac{1}{12\pi} \). Since this is not rational, \( x_3(n) \) is not periodic.

Some values for each of these signals are shown in the table below. The asterisks in the table cells show when the initial value repeats.
### Discrete-Time Signals

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_1(n)$</th>
<th>$x_2(n)$</th>
<th>$x_3(n)$</th>
</tr>
</thead>
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<td>* 0</td>
<td>* 0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.7071</td>
<td>0.16589</td>
</tr>
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</tr>
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<td>0.7071</td>
<td>0.47942</td>
</tr>
<tr>
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<td>* 0</td>
<td>0</td>
<td>0.61837</td>
</tr>
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<td>1</td>
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<td>0.74017</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>-0.7071</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>* 0</td>
<td>* 0</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

From this, we see that not all DT exponentials/sinusoids are periodic in time.

We can summarize periodicity in time, by contrasting DT with CT.

- In CT, $e^{j\omega_o t}$ is always periodic, with $T_o = 2\pi/|\omega_o|$.
- In DT, $e^{j\omega_o n}$ might be periodic, only when $\omega_o/2\pi = k/N = k_o/N_o$ is rational.

#### Example 3.2: Sampling a CT Sinusoid to DT

Say we have a CT sinusoid.

$$x_c(t) = \sin\left(\frac{3\pi}{4}t\right)$$

This CT signal has a fundamental period $T_o = 8/3$.

Let’s sample this to DT using $T_s = 1$. (Note that $T_s < T_o/2$, so we meet requirements of sampling theorem).

We get a DT signal as follows.

$$x_d(n) = x_c(t)|_{t=nT_s=n} = \sin\left(\frac{3\pi}{4}n\right)$$

So $\omega_o = \frac{3\pi}{4}$, and $\frac{\omega_o}{2\pi} = \frac{3\pi}{4}/2\pi = \frac{3}{8}$ which gives us $N_o = 8$ (which is equal to $3T_o$).

#### 3.5.2 Property 2: Periodicity in Frequency ($\omega$)

In CT, all sinusoids $\sin\omega_o t$ and exponentials $e^{j\omega_o t}$ are distinguishable by $\omega_o$.

For example, $e^{j\omega_1 t} \neq e^{j\omega_2 t}$ unless $\omega_1 = \omega_2$. (Note that $\neq$ means not equal $\forall t \in \mathbb{R}$).
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Also, the rate of oscillation increases at \(|\omega_o|\) increases.

In DT, however, exponentials are not all distinguishable.

For example, consider \(e^{j\omega_0 n}\) and \(e^{j(\omega_0 + 2\pi k)n}\).

\[
e^{j(\omega_0 + 2\pi k)n} = e^{j\omega_0 n} \cdot e^{j2\pi kn} \quad \text{for} \quad \forall k, n \in \mathbb{Z}
\]

\[
e^{j\omega_0 n} \quad \text{since (by Euler’s relation)} \quad e^{j2\pi kn} = 1 \quad \text{for} \quad \forall k, n \in \mathbb{Z}
\]

So \(\omega_o\) and \(\omega_o + 2\pi k\) are indistinguishable frequencies in DT. (This is why aliasing occurs – with insufficient sampling, the CT frequencies mapped to DT are indistinguishable).

We speak of frequencies in a range of \(2\pi\), since on frequencies between \(\omega_o\) and \(\omega_o + 2\pi\) can be distinguished. By convention, we usually use the range \([-\pi, \pi)\) or sometimes \([0, 2\pi)\). (Recall that the notation means that the square-bracket end includes that value; the round-bracket end of the range is not included).

3.5.3 Property 3: Harmonically Related Exponentials and Sinusoids

In CT, \(x(t) = e^{j\omega_0 t}\) with fundamental period \(T_o = 2\pi/|\omega_o|\) has a harmonically related family

\[
\Phi_k(t) = e^{j\omega_o t} \quad \text{where} \quad k \in \mathbb{Z}
\]

all having period \(T_o\) and used to form the CT Fourier series and other periodic CT signals.

In DT, consider an exponential of form \(\Phi_1(n) = e^{j2\pi n}\) having \(\omega_o = 2\pi/N\) for \(N \in \mathbb{Z}^+\) and thus \(\frac{\omega_o}{2\pi} = \frac{1}{N}\) and so \(N_o = N\).

Such an exponential has a harmonically related family

\[
\Phi_1(n) = e^{j\frac{2\pi}{N} kn} \quad \text{with} \quad \omega_k = \frac{2\pi}{N} k
\]

and so \(\frac{\omega_k}{2\pi} = \frac{k}{N}\) is rational.

But, rather than the infinite family as in CT, only a finite number of \(\Phi_k(n)\) are distinguishable for

\[
\frac{2\pi k}{N} \in [0, 2\pi) \quad \text{only for} \quad k = 1, 2, 3, ..., N - 1
\]

so there are \(N\) members of the family.

This finite family is used to represent other periodic DT signals with fundamental period \(N\). The Discrete-Time Fourier Series (DTFS) is therefore a finite sum, and is thus computable. (We can compute it via the Fast Fourier Transform (FFT) algorithm very efficiently – \(O(N \log N)\)).
Summary of Properties of DT Exponentials and Sinusoids

- DT sinusoids and exponentials are periodic in time iff \( \frac{\omega_o}{2\pi} = \frac{k}{N} \) is rational.
- DT sinusoids and exponentials are periodic in frequency since \( e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi k)n} \). So, starting from \( \omega_o \), only \([\omega_o, \omega_o + 2\pi)\) are distinguishable frequencies in DT. We usually use ranges \([0, 2\pi)\) or \([-\pi, \pi)\).

\[
x_c(t) = \sin \omega_o t \xrightarrow{T_s} x_d(N) = \sin (\omega_o T_s n) = \sin \left(2\pi \frac{\omega_o}{\omega_s} n\right)
\]

If \( \omega_s > 2\omega_o \) we have no aliasing, and so our CT non-alias range maps to the DT distinguishable base range.

\[
\begin{bmatrix}
-\omega_s/2 & \omega_s/2
\end{bmatrix} \xrightarrow{T_s} [-\pi, \pi)
\]

If \( \omega_s \neq 2\omega_o \) we have aliasing, and so our DT frequency is not in the distinguishable base range (\( \omega_d \not\in [-\pi, \pi) \)).

\[
\begin{bmatrix}
\omega_s/2 & 3\omega_s/2
\end{bmatrix} \xrightarrow{\omega_d = 2\pi \omega_o/\omega_s} [\pi, 3\pi)
\]

In other words \([\omega_s/2, 3\omega_s/2]\) maps to \([\pi, 3\pi]\), which is indistinguishable from \([-\pi, \pi]\), and thus \([\omega_s/2, 3\omega_s/2]\) aliases to \([-\omega_s/2, \omega_s/2]\).