FREQUENCY RESPONSE REVIEW

Can define a frequency response for a linear time-invariant system (LTI).

\[ U(t) \rightarrow \text{SYSTEM} \rightarrow Y(t) \]

Linear if \( U_1(t) \) results in \( Y_1(t) \)

\( U_2(t) \) results in \( Y_2(t) \)

Then linear if and only if

\( U_1(t) + U_2(t) \) results in \( Y_1(t) + Y_2(t) \)

For any \( U_1(t) \), \( U_2(t) \)

TIME IN Variant

Time invariant if and only if

If \( U_1(t) \) results in \( Y_1(t) \)

Then \( U_1(t - T) \) results in \( Y_1(t - T) \)

For any \( U_1(t) \) and \( T \)
FREQUENCY RESPONSE

LET LTI SYSTEM HAVE TRANSFER FUNCTION \( T(s) \)

\[
\begin{array}{c}
\text{u}(t) \quad \text{T}(s) \quad \text{y}(t)
\end{array}
\]

FREQUENCY RESPONSE IS GIVEN BY

LETTING \( s = j\omega \) WHERE "\( \omega \)" IS ANGULAR FREQUENCY IN RAD/\( s \)

\[
\omega = 2\pi f \quad \text{WHERE} \quad f \text{ is in \( \frac{\text{cycles}}{s} \) or Hz}
\]

\[
\text{u}(t) = A_0 \sin (\omega t) = A_0 \sin (2\pi ft)
\]

\[
T_p = \frac{1}{f} = \frac{2\pi}{\omega}
\]

FOR \( \omega = 6.28 \times 10^3 \) RAD/\( s \)

\[
f = \frac{\omega}{2\pi} = 1 \text{ kHz}
\]

\[
T_p = 1 \text{ ms}
\]

\[
\text{y}(t) = A_0 \sin (\omega t + \phi)
\]

\[
T_p = \frac{1}{f}
\]
For LTI

If input is sinusoidal with freq f

Then output is also sinusoidal

with freq f but different amplitude + phase.

\[
\frac{A_q}{A_p} = |T(iw)| \quad \text{MAGNITUDE RESPONSE}
\]

\[
\phi = \angle T(iw)
\]

\(T(iw)\) is a complex number where

The \(|T(iw)|\) is magnitude response

and \(\angle T(iw)\) is phase response.
**Complex Numbers**

- **A**: $|A| = 1$, $\angle A = 0^\circ$
- **B**: $|B| = 1$, $\angle B = 180^\circ$
- **C**: $|C| = 1$, $\angle C = 90^\circ$
- **D**: $|D| = \sqrt{2}$, $\angle D = 45^\circ$

If $z = a + ib$, then $z$ is complex with $a, b$ real.

- $|z| = (a^2 + b^2)^{1/2}$
- $\angle z = \tan^{-1}\left(\frac{b}{a}\right)$ if $a > 0$
- $\angle z = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ if $a < 0$
RATIO OF COMPLEX NUMBERS

\[ Z = \frac{Z_1}{Z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{|Z_1| e^{j<z_1}}{|Z_2| e^{j<z_2}} \]

\[ |Z| = \frac{|Z_1|}{|Z_2|} = \left( \frac{a_1^2 + b_1^2}{a_2^2 + b_2^2} \right)^{\frac{1}{2}} \]

\[ \angle Z = \angle z_1 - \angle z_2 \]

\[ = \tan^{-1} \left( \frac{b_1}{a_1} \right) - \tan^{-1} \left( \frac{b_2}{a_2} \right) \quad \text{if} \quad a_1 > 0 \quad \text{and} \quad a_2 > 0 \]
Ohm's Law (Impedance)

\[ I = \frac{V}{Z} \]

Resistor of size \( R \) \( \Rightarrow \) \( Z = R \)

Capacitor of size \( C \) \( \Rightarrow \) \( Z = \frac{1}{sC} \)

Inductor of size \( L \) \( \Rightarrow \) \( Z = sL \)

"S" is Laplace Transform Variable

We let \( S = j\omega \) to evaluate what happens at frequency \( \omega \) (Omega)

Why is there the complex variable "j"??
\[ I_c = I_s \]

\[ I_s = A_p \sin (\omega t) = A_p \sin (2\pi f t) \]

\[ T = \frac{1}{f} \]

**Example**  
\[ A_p = 1 \text{ mA} \]  
\[ \omega = 6.28 \times 3 \text{ rad/s} \]  
\[ f = \frac{\omega}{2\pi} = 1 \text{ kHz} \]  
\[ T = 1 \text{ ms} \]

It turns out that voltage \( V_c \) is 90° out of phase with \( I_c \). Hence the need for "j"
In freq domain, transfer function can be found

\[ T(s) = \frac{V_c(s)}{I_c(s)} \Rightarrow I_c(s) = \frac{V_c(s)}{(j\omega C)} \]

\[ T(s) = \frac{V_c(s)}{I_c(s)} = \frac{1}{j\omega C} \]

Magnitude response \[ |T(j\omega)| = \left| \frac{1}{j\omega C} \right| = \frac{1}{\omega C} \]

Phase response \[ \angle T(j\omega) = \angle \left( \frac{1}{j\omega C} \right) \]

\[ \angle T(j\omega) = \angle (1) - \angle (j\omega C) = 0^\circ - 90^\circ = -90^\circ \]
If $C = 1 \text{mF}$

$$\frac{A_p}{\omega C} = \frac{1 \text{mA}}{(6.28 \times 10^3)(1 \text{mF})} = 0.159 \text{ V}$$
TRANSFER FUNCTION OF LTI

- RESTRICT OURSELVES TO

- REAL-VALUED IMPULSE RESPONSE
- CIRCUITS WITH LUMPED ELEMENTS
- RESISTORS, CAPACITORS, INDUCTORS,
  INDEPENDENT + DEPENDENT VOLTAGE
  + CURRENT SOURCES

\[ T(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots + b_n s^n} \tag{1} \]

OR

\[ T(s) = \left(\frac{a_m}{b_n}\right) \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + w_1)(s + w_2) \ldots (s + w_n)} \]

FOR STABILITY \( n \geq m \)

1. POLYNOMIAL FORM
2. ROOT FORM SHOWS ZEROS \(-z_i\)
   POLES \(-w_i\)

IN GENERAL, \(z_i\) & \(w_i\) CAN BE COMPLEX VALUES & OCCUR IN COMPLEX CONJUGATE PAIRS

HOWEVER, FOR THIS FREQUENCY ANALYSIS SECTION ALL \(z_i\) & \(w_i\) REAL VALUED
IN OTHER WORDS, ALL POLES & ZEROS WILL BE ON THE REAL-AXIS.

ALSO FOR A STABLE SYSTEM, ALL POLES WILL BE IN NEGATIVE HALF-PLANE (AND HENCE ON NEGATIVE REAL-AXIS HERE).

ZEROS WHERE $T(s) = 0$

Zero at $s = z_1 \Rightarrow T(z_1) = 0$

POLES WHERE $T(s) \rightarrow \infty$

Pole at $s = w_i \Rightarrow T(w_i) \rightarrow \infty$

**Ex.** $T(s) = \frac{1}{s+2}$  NO ZEROS

Pole $w_1 = -2$

**Ex.** $T(s) = \frac{s}{s+3}$  ZERO $z_1 = 0$

Pole $w_1 = -3$

$T(s) = \frac{s(s-2)}{(s+1)(s+3)}$  ZEROS: $z_1 = 0$ $z_2 = 2$

POLES: $p_1 = -1$ $p_2 = -3$
\[ T(s) = \left(\frac{\alpha_m}{b_n}\right) \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+w_1)(s+w_2)\ldots(s+w_n)} \]

**Strictly Speaking:** Poles at \(-w_i\) and Zeros at \(-z_i\)

**However:** (For Real Axis Poles/Zeros)

Often say Pole Frequency at \(w_i\)
Or Zero Frequency at \(z_i\)

Since if we only consider pole (or zero) at \(w_i\)

\(T(s)\) is reduced by \(\frac{1}{w_i}\) at \(T(jw_i)\)

**Example:**

\[ T(s) = \frac{1}{s+1} \]

\(T(0) = 1\)

\(T(\infty) = 0\)

\(T(j1) = \frac{1}{\sqrt{2}}\)

So for \(w = 1\) Rad/s

\[ |T(j1)| = \frac{1}{\sqrt{2}} T(0) \text{ or 3 dB down} \]

![Graph](image)
\[ T(s) = \frac{s+2}{s+100} \]

\[ T(0) = 0.02 \quad T(j\infty) = 1 \]

\[ |T(j\omega)| \approx \sqrt{2} |T(0)| \quad 3\text{dB INCREASE} \]

\[ |T(j100)| \approx \frac{1}{\sqrt{2}} |T(j\infty)| \quad 3\text{dB DECREASE} \]

ZERO AT 2 RAD/SEC

POLE AT 100 RAD/SEC
ALTERNATE ROOT FORM

\[
T(s) = \frac{a_m}{b_n} \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + w_1)(s + w_2) \ldots (s + w_n)}
\]

\[
T(j\omega) = \begin{cases} 
\frac{a_m}{b_n} & \text{if } m = n \\
0 & \text{if } m < n
\end{cases}
\]

\[
T(0) = \frac{(a_m)}{(b_n)} \frac{z_1 z_2 \ldots z_m}{w_1 w_2 \ldots w_n}
\]

ALTERNATIVELY \( T(s) \) WRITTEN AS

\[
T(s) = k_{dc} \frac{(1 + s z_1)(1 + s z_2) \ldots (1 + s z_m)}{(1 + s w_1)(1 + s w_2) \ldots (1 + s w_n)}
\]

\[
T(j\omega) = 0 \quad \text{if } m < n
\]

\[
(\text{complicated if } m = n)
\]

\[
T(0) = k_{dc}
\]