An Analytical Method for the 3D-Location Estimation of Circular Features for an Active-Vision System

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ABSTRACT

In the context of a new 3D-object-recognition system, currently under development at the University of Toronto, the problem of location estimation (position and orientation) of a circular-marker feature is a major one. In this paper, a closed-form analytical solution to the circular-feature problem is presented. Compared to previous solution methods, it has the following advantages: (1) It is mathematically simpler; (2) It provides solutions for the case in which there does not exist a priori knowledge concerning the radius of a marker; (3) It can be extended and applied to general quadratic surfaces; and (4) From a geometrical point of view, the solution method is clearer. The developed method was applied to a set of circles located on a calibration plate, whose locations were known with respect to a reference frame. The camera was calibrated prior to the application of the method. Furthermore, since various distortion factors had to be compensated in order to obtain accurate estimates of the parameters of the imaged circle - an ellipse - with respect to the camera's image frame, a sequential compensation procedure was applied to the input grey-level image. The experimental results obtained show the validity of the method developed.

1. Introduction

In a proposed method for the recognition of pre-marked 3D objects using circular markers, one of the first problems to be solved was surface-normal estimation of a circular marker feature [1]. Initially, a method based purely on geometrical reasoning was developed [2]. Later, in the process of experimentation, this method was found not to be general but to provided acceptable solutions only for a set of specific camera-marker configurations. Motivated by this experimental result to initiate a more in-depth study of the geometrical properties of ellipses and their perspective projections, it was found that our 3D-geometrical representation of the problem was indeed not general, but represents a set of special cases. Thus, we attempted to expand the same approach (that of geometrical reasoning) to solve the general case. Though we were able to solve part of the problem, it was found to be very difficult to solve the total problem by applying only geometrical reasoning. Thus, it was decided to use the full force of 3D analytical geometry in the solution. Thereby a method has been developed to solve the problem in its general form.

This problem, the estimation of location of a circular marker (feature), can be classified as a special case of a more general problem which, in the machine-vision literature, is referred to as feature-based-3D-location estimation of objects in a scene. The main body of literature dealing with this general problem is concerned with developing mathematical methods for the estimation of an object's location based on **point** features. Estimation of the object's location based on **line** features has also been studied, but to a lesser extent. As well, for quadratic curves, a general method has been developed [2]; this method is iterative and requires an initial estimation. For a circular feature, representing in particular a special case of quadratic-curved features, an iterative method has been developed [4]. Closed-form solutions for circular features have also been developed based on linear algebra [5] and vector algebra [6]; but, these methods are mathematically complex and, being based on linear algebra (and vector algebra), do not provide a geometrical representation of the problem, nor a geometrical interpretation of the resulting solutions.

Recently, a closed-form mathematical solution, based on 3-D analytical geometry of circular and spherical features, has been published [7]. This solution method has several advantages: it is a closed-form solution; it gives only the necessary number of solutions (with no redundant solutions); it uses simple mathematics involving 3D-analytic geometry; and, finally, it is geometrically intuitive.

In this paper, we present another mathematical solution to the circular feature problem based on 3D-analytic geometry. It provides all the above-mentioned advantages compared to all previous methods. But, as well, it has the following advantages: (1) It is, mathematically, still simpler; (2) It provides solutions for the case in which there does not exist a-priori knowledge concerning the radius of the marker; (3) It can be extended and applied to general quadratic surfaces due to its more general formulation ; (4) The solution method is clearer from a geometrical point of view.

2. Analytical Formulation of the Problem and an Iterative Solution

2.1 Problem statement

Given the effective focal length of a camera, and the five basic parameters of an ellipse in the image-coordinate frame (the perspective projection of a circular marker in 3D object space onto a 2D image space), it is required to estimate the circular-marker's surface normal with respect to the camera frame.

This problem is equivalent to the following one: Given the equation of the base of a cone and its vertex with respect to a reference frame, determine the orientation of a plane (with respect to the same reference frame) which intersects the cone and generates a circular curve.

At first, an attempt was made to find a solution based on purely geometrical reasoning. It was found that this was a very difficult task. Subsequently, an attempt was made to find a solution by applying 3D analytical geometry. It was found that solving the problem in its general form would lead to a set of two highly nonlinear equations, the solutions of which would need nonlinear optimization (implying an iterative process). Furthermore, this process would produce at least eight sets of solutions, though there exist, at most, two acceptable sets of solutions. Through further analysis, a set of transformations were developed that simplify the solution of the problem and provide analytically the only acceptable sets of solutions.

2.2 The equation of a cone with a given curve as its base

The first step in the proposed general solution is to find the equation of the cone whose vertex is the point (α, β, γ) and whose base is defined as (in our case, an ellipse):

$$\begin{cases} F(x,y) \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + d' = 0 \\ z = 0 . \end{cases} \quad (1)$$

This problem is schematically shown in Figure 1 for $\alpha = 0$, $\beta = 0$, and $\gamma = -e$, where e is the effective focal length of the camera.

It has been proven that the equation of the cone can be expressed as [8]:

$$z^2 F(\alpha,\beta) - z\gamma \left[x \frac{\partial F}{\partial \alpha} + y \frac{\partial F}{\partial \beta} + t \frac{\partial F}{\partial t} \right] + \gamma^2 F(x,y) = 0 . \quad (2)$$

In this equation, t is an auxiliary variable by which $F(x,y)$ is made homogeneous $(F(\alpha,\beta,t))$. The term $t\,(\partial F/\partial t)$ is calculated by first taking the derivative of the homogeneous equation with respect to t and then equating t to unity. The general form of the equation of a cone that is derived using the above equation is as follows:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 . \quad (3)$$

2.3 Circular section of a cone

It can be proven that all parallel planar sections of a conicoid are similar and similarly situated conics. Thus, an intersection plane can be defined by $lx + my + nz = 0$. Therefore, the problem of finding the coefficients of the equation of a "circular-section" plane can be expressed mathematically as: finding l, m, and n such that the intersection of the following surfaces is a circle:

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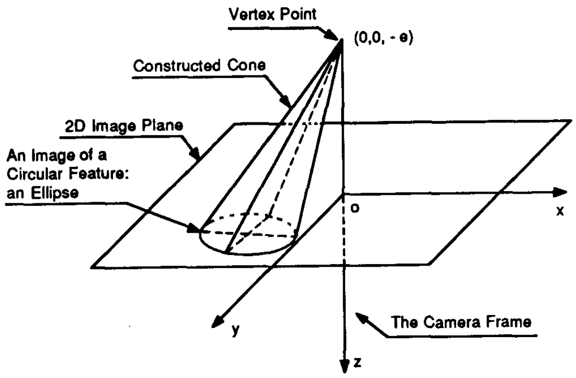

Figure 1. Schematic representation of the problem for a circular feature.

$$\begin{cases} ax^2 + by^2 + cz^2 + 2fyz + 2gzx + \\ \qquad 2hxy + 2ux + 2vy + 2wz + d = 0 \\ lx + my + nz = 0, \end{cases} \tag{4}$$

where $l^2 + m^2 + n^2 = 1$. Having found the coefficients of the equation of the plane, the surface-normal direction numbers (l',m',n') can be estimated by:

$$\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}. \tag{5}$$

Now, in order to find the equation of the intersection curve of the above two surfaces (4), the following rotational transformation can be used:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{m}{\sqrt{l^2+m^2}} & -\frac{ln}{\sqrt{l^2+m^2}} & l \\ \frac{l}{\sqrt{l^2+m^2}} & -\frac{mn}{\sqrt{l^2+m^2}} & m \\ 0 & \sqrt{l^2+m^2} & n \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \tag{6}$$

This transformation is defined such that the new z-axis (i.e., z') would be normal to the plane $lx + my + nz = 0$. In this case the equation of the intersecting plane with respect to the new frame is $z' = 0$. Thus the equation of the intersection curve with respect to the new frame would be:

$$\begin{aligned} &[\, am^2 + bl^2 - 2hlm \,]\, x'^2 + [\, al^2n^2 + bm^2n^2 + c(l^2+m^2)^2 + \\ &2hlmn^2 - 2gln(l^2+m^2) - 2fmn(l^2+m^2) \,]\, y'^2 + [\, 2almn - \\ &2blmn - 2hl^2n + 2hm^2n - 2gm(l^2+m^2) + 2lf(l^2+m^2) \,]\, x'y' + \\ &2[\, -um\sqrt{l^2+m^2} + vl\,\sqrt{l^2+m^2} \,]\, x' + \\ &2[\, -uln\sqrt{l^2+m^2} - vmn\,\sqrt{l^2+m^2} + w(l^2+m^2)^{3/2} \,]\, y' + \\ &(l^2+m^2)d = 0. \end{aligned} \tag{7}$$

Note that if $z' = \text{constant} \neq 0$, then the values for coefficients of x', y' and the constant term would change.

It can be proven that the necessary and sufficient conditions for which a general quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{8}$$

represents a circle would be,

$$A = C, \quad B = 0. \tag{9}$$

Thus, for equation (7), the following two conditions must exist:

$$(a-b)\, lmn + f\,(l^2+m^2)\, l - g\,(l^2+m^2)\, m - hl^2n + hm^2n = 0 \tag{10}$$

$$am^2 + bl^2 - 2hlm = al^2n^2 + bm^2n^2 + c(l^2+m^2)^2 - 2f(l^2+m^2)\,mn - 2g(l^2+m^2)\,ln + 2hlmn^2$$

Considering these two equations (10) together with the third equation $l^2 + m^2 + n^2 = 1$, three nonlinear equations with three unknowns l, m, and n would be obtained. From the first equation of (10), n can be expressed in term of the other two unknowns, based on which, two nonlinear equations (of higher degree) with two unknowns l, and m would be obtained:

$$\begin{aligned} &am^2 + bl^2 - 2hlm = c(l^2+m^2)^2 + \\ &(al^2 + bm^2 + 2hlm)\left[\frac{-f(l^2+m^2)l + g(l^2+m^2)m}{alm - blm - hl^2 + hm^2}\right]^2 - \\ &2\,[\, f(l^2+m^2)m + g(l^2+m^2)l \,]\left[\frac{-f(l^2+m^2)l + g(l^2+m^2)m}{alm - blm - hl^2 + hm^2}\right] \end{aligned} \tag{11}$$

$$l^2 + m^2 - 1 = -\left[\frac{-f(l^2+m^2)l + g(l^2+m^2)m}{alm - blm - hl^2 + hm^2}\right]^2. \tag{12}$$

Variables l and m are of degree eight in the first equation and of degree six in the second equation. To solve these two nonlinear equations, (iterative) nonlinear optimization techniques can be used. It is apparent that this is not an efficient way to solve this problem. Furthermore, the problem of acceptable solutions would remain, since analytically, these equations can have up to eight sets of solutions, though intuitively it is known that there exist at most two physical solution sets.

In the following section, an alternative method is presented, by which an analytical solution is possible. This method is based on a reduction of the general equation of conicoids.

3. An Analytical Solution of the Problem

3.1 Reduction of the general equation of conicoids

In this section, a method is presented to reduce the general equation of conicoids, of the following form [9]:

$$\begin{aligned} ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + \\ 2ux + 2vy + 2wz + d = 0 \end{aligned} \tag{13}$$

to a more compact form of

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = \mu. \tag{14}$$

In essence, this reduction is based on a transformation consisting first of a rotation and then a translation of the xyz-frame.

3.1.1 Rotational transformation

Let the rotational transformation be defined as,

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \tag{15}$$

and consider the homogeneous equation of conicoids as,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \tag{16}$$

The problem is to find the elements of a rotational transformation such that (16) reduces to the following form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 = 0. \tag{17}$$

It has been proven in [9] that if λ_1, λ_2, and λ_3 are the roots of the equation (called the discriminating cubic)

$$\begin{aligned} \lambda^3 - \lambda^2\,(a+b+c) + \lambda\,(bc + ca + ab - f^2 - g^2 - h^2) - \\ (abc + 2fgh - af^2 - bg^2 - ch^2) = 0, \end{aligned} \tag{18}$$

then the elements of the rotational transformation would be obtained from the following equations:

$$\frac{al_i + hm_i + gn_i}{l_i} = \frac{hl_i + bm_i + fn_i}{m_i} = \frac{gl_i + fm_i + cn_i}{n_i} = \lambda_i, \quad i = 1,2,3. \tag{19}$$

3.1.2 Translational transformation

Through the rotational transformation (15), the general equation of conicoids (13) would reduce to the following form:

$$\begin{aligned} &\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2\,(ul_1 + vm_1 + wn_1)\,x' + \\ &2\,(ul_2 + vm_2 + wn_2)\,y' + 2\,(ul_3 + vm_3 + wn_3)\,z' + d = 0. \end{aligned} \tag{20}$$

It can be proven that the translational transformation required to reduce equation (20) to equation (14) would be:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -(ul_1 + vm_1 + wn_1)/\lambda_1 \\ 0 & 1 & 0 & -(ul_2 + vm_2 + wn_2)/\lambda_2 \\ 0 & 0 & 1 & -(ul_3 + vm_3 + wn_3)/\lambda_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}. \tag{21}$$

The following set of equations would be obtained when the two transformations are combined,

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -(ul_1 + vm_1 + wn_1)/\lambda_1 \\ 0 & 1 & 0 & -(ul_2 + vm_2 + wn_2)/\lambda_2 \\ 0 & 0 & 1 & -(ul_3 + vm_3 + wn_3)/\lambda_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}. \tag{22}$$

Thus, through the general transformation (22), the general equation of conicoids (13) would reduce to what is referred to as the equation of central conicoids:

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = \mu. \tag{23}$$

If the general transformation (22) is applied to the general equation of a cone, it can be proven that its equation would be in the form of equation (23), where $\mu = 0$ and two of the three coefficients would be always positive and one always negative [9]. If positive values are assigned to λ_1 and λ_2 and a negative value to λ_3, then the principal axis of the central cone would be the Z-axis of the XYZ-frame. This case is shown in Figure 2. The intersections of parallel planes $Z = k$ and the central cone would then generally be ellipses of different sizes:

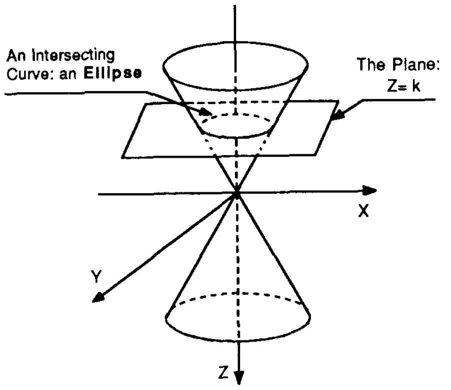

Figure 2. The central form of a cone: an **elliptical** right cone.

$$\frac{X^2}{(k\sqrt{(-\lambda_3/\lambda_1)})^2} + \frac{Y^2}{(k\sqrt{(-\lambda_3/\lambda_2)})^2} = 1\,.$$

3.2 Circular section of a central cone

In this section, the problem that was presented in section 2 would be solved analytically, this time by considering the equation of a cone in its central form. Thus, it is required to find the coefficients of the equation of a plane (with respect to the XYZ-frame)

$$l\,X + m\,Y + n\,Z = p \tag{24}$$

such that its intersection with a central cone (let the first two coefficients be positive and the third one be negative)

$$a'X^2 + b'Y^2 + c'Z^2 = 0 \tag{25}$$

would be a circle.

Applying the rotational transformation (6), the equation of the plane would be of the form $Z' = p$ and thus the equation of the central cone would be of the following form:

$$\begin{aligned}&\left[\frac{a'm^2}{l^2+m^2} + \frac{b'l^2}{l^2+m^2}\right]X'^2 + \\ &\left[\frac{a'l^2n^2}{l^2+m^2} + \frac{b'm^2n^2}{l^2+m^2} + c'\,(l^2+m^2)\right]Y'^2 + \\ &\left[\frac{2a'lmn}{l^2+m^2} - \frac{2b'lmn}{l^2+m^2}\right]X'Y' + \\ &p\left[-\frac{2a'lm}{\sqrt{l^2+m^2}} + \frac{2b'lm}{\sqrt{l^2+m^2}}\right]X' + \\ &p\left[-\frac{2a'l^2n}{\sqrt{l^2+m^2}} - \frac{2b'm^2n}{\sqrt{l^2+m^2}} + 2c'n\sqrt{l^2+m^2}\right]Y' + \\ &p^2\,[\,a'l^2 + b'm^2 + c'n^2\,] = 0\,.\end{aligned} \tag{26}$$

To produce a circular curve of intersection, the following conditions must be satisfied (applying (9)):

$$2(a' - b')\,\frac{lmn}{l^2+m^2} = 0 \tag{27}$$

$$\frac{1}{l^2+m^2}\,(a'm^2 + b'l^2) = \frac{a'l^2n^2}{l^2+m^2} + \frac{b'm^2n^2}{l^2+m^2} + c'(l^2+m^2). \tag{28}$$

Knowing that $l^2 + m^2 \neq 0$, (28) is simplified as,

$$(a'l^2 + b'm^2)n^2 + c'(l^2+m^2)^2 = a'm^2 + b'l^2\,. \tag{29}$$

Furthermore, the general relationship between coefficients exists:

$$l^2 + m^2 + n^2 = 1\,. \tag{30}$$

Thus, there exist three equations (27, 29, and 30) and three unknowns l, m, and n.

Considering equation (27), four possible cases exist.

Case (I): $l = 0$

Based on equations (27), (29), and (30), the following solutions are derived:

$$\begin{cases} n = \pm\sqrt{\dfrac{a'-c'}{b'-c'}} \\ m = \pm\sqrt{\dfrac{b'-a'}{b'-c'}} \\ l = 0\,. \end{cases}$$

The above solutions must be checked to determine whether they are acceptable ones. If it is assumed that the principal axis of the central cone is the Z'-axis, then:

$$a' > 0,\quad b' > 0,\quad c' < 0, \tag{31}$$

which leads to the following inequalities:

$$a' - c' > 0,\quad b' - c' > 0. \tag{32}$$

Based on these inequalities, one can check the solutions. The solutions for n are acceptable since the expression $\frac{a'-c'}{b'-c'}$, is positive. For m, the solutions would be acceptable if $b' > a'$.

If it is assumed that $b' > a'$, then there exist four solutions to the problem. But these are four symmetrical solutions with respect to the origin of $X'Y'Z'$-frame and consequently represent only two unique solutions. If one takes the solutions on the positive section of Z'-axis, then the two solutions would be:

$$\begin{cases} n = +\sqrt{\dfrac{a'-c'}{b'-c'}} \\ m = \pm\sqrt{\dfrac{b'-a'}{b'-c'}} \\ l = 0\,. \end{cases} \tag{33}$$

Case (II): $m = 0$

Following the same arguments for case (I), two solutions can be derived, which would be acceptable only if $a' > b'$:

$$\begin{cases} n = +\sqrt{\dfrac{b'-c'}{a'-c'}} \\ m = 0 \\ l = \pm\sqrt{\dfrac{a'-b'}{a'-c'}}\,. \end{cases} \tag{34}$$

Case (III): $n = 0$

In this case, the following solutions for l and m are derived:

$$l = \pm\sqrt{\frac{c'-a'}{b'-a'}}$$

$$m = \pm\sqrt{\frac{b'-c'}{b'-a'}}\,.$$

However for the solutions of l to be acceptable, one must have $b' < a'$; and for the solutions of m to be acceptable, on the other hand, one must have $b' > a'$. Thus, it can be claimed that acceptable solutions does not exist for this case.

Case (IV): $a' = b'$

This is a special case, since it imposes a constraint on the coefficients of the equation of a central cone. In this case, the equation of the central cone represents a right circular cone (which implies that the central surface normal of the circular feature passes through the origin of the camera frame), and thus, any plane $Z' = p$ intersects it and generates a circular intersection curve. Thus, there exists only one solution :

$$\begin{cases} n = 1 \\ m = 0 \\ l = 0\,. \end{cases} \tag{35}$$

This solution can also be derived directly from equations (27), (29), and (30).

In conclusion, one can state that, generally, there exist two unique solutions to the problem. Under special conditions, these two solutions reduce to a unique solution. The direction cosines of the surface normal of the circular feature can be estimated using relation (5), once the coefficients of the equation of the desired plane have been estimated.

3.3 The computation procedure for surface-normal estimation

The computation procedure for the proposed analytical method consists of the following steps:

1. *Derivation of the general equation of the cone:* One can derive the equation of the cone and thus estimate the coefficients a, b, c, f, g, h, u, v, w, and d, based on the knowledge of the equation of an ellipse in the image coordinate frame and the effective focal length of the camera, and by applying equation (2).

2. *Reduction of the equation of the cone:* One can determine the coefficients λ_1, λ_2, and λ_3 in equation (23), by solving the discriminating cubic equation (18). These are determined such that the coefficients of X and Y would be positive. Using (19), one can estimate the elements of the rotational transformation matrix, and using (21), one can estimate the elements of the translational transformation matrix. Thus, the general transformation matrix (22) can be obtained.
3. *Estimation of the coefficients of the equation of the desired plane:* Having estimated the coefficients of the central cone in part 2 (λ_i in (23) or a', b' and c' in (25)), three possible cases occur: (1) $a' < b'$, for which the solutions would be (33); (2) $a' > b'$, for which the solutions would be (34); and $a' = b'$, for which the solutions would be (35).
4. *Estimation of the direction cosines of the surface normal with respect to the camera frame:* Since only the orientation of the circular marker's surface normal is needed, one must only consider the rotational transformation involved in this method. Thus, if T_1 is the transformation between the $x'y'z'$-frame and the xyz-frame, then the coefficients of the equation of the desired plane with respect to the xyz-frame can be estimated by applying the rotational transformation T_1. Following which, the direction cosines of the surface normal are estimated using relation (5).

4. Estimation of the Position of a Circular Feature

The position of a circle is defined by its center coordinates, (x_o, y_o, z_o), with respect to the xyz camera frame. Depending on whether the radius of the circle is known or it is not, two different methods can be applied.

4.1 Case (I): Radius is known

Based on the knowledge of the orientation of a circle and its radius, one can solve the position problem as follows. The problem is simplified by first solving it in the $X'Y'Z'$-frame and then applying the total transformation, $T = T_1T_2T_3$ where

$$X'Y'Z' \xrightarrow[T_3]{\text{Rotation}} XYZ \xrightarrow[T_2]{\text{Translation}} x'y'z' \xrightarrow[T_1]{\text{Rotation}} xyz,$$

to estimate the position with respect to the camera frame (xyz). This is possible only because the length of a radius is invariant with respect to the rotational and translational transformations of a frame.

The desired plane of intersection is defined as $Z' = p$ with respect to the $X'Y'Z'$-frame. The elements of the transformation (6) are already known (the coefficients of the equation of the desired plane l, m, n are known). Thus if one defines the transformation as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}, \tag{36}$$

then the equation of the circle with respect to the $X'Y'Z'$-frame (26) would be ($Z' = p$):

$$(a'l_1^2 + b'l_2^2 + c'l_3^2)X'^2 + (a'm_1^2 + b'm_2^2 + c'm_3^2)Y'^2 + 2(a'l_1m_1 + b'l_2m_2 + c'l_3m_3)X'Y' + 2p(a'l_1n_1 + b'l_2n_2 + c'l_3n_3)X' + 2p(a'm_1n_1 + b'm_2n_2 + c'm_3n_3)Y' + p^2(a'n_1^2 + b'n_2^2 + c'n_3^2) = 0. \tag{37}$$

As it was noted earlier, the coefficient of the $X'Y'$-term must be zero and the coefficients of X'^2 and Y'^2 must be equal. Now, if

$$A \equiv (a'l_1^2 + b'l_2^2 + c'l_3^2)$$
$$B \equiv (a'l_1n_1 + b'l_2n_2 + c'l_3n_3)$$
$$C \equiv (a'm_1n_1 + b'm_2n_2 + c'm_3n_3)$$
$$D \equiv (a'n_1^2 + b'n_2^2 + c'n_3^2)$$

then the equation of the circle would become (in its standard form):

$$(X' + \frac{pB}{A})^2 + (Y' + \frac{pC}{A})^2 = \frac{p^2B^2}{A^2} + \frac{p^2C^2}{A^2} - \frac{p^2D}{A}. \tag{38}$$

However, the radius (r) is known, and therefore one can estimate the value of the parameter p from the following equation:

$$p = \pm \frac{A\,r}{\sqrt{B^2 + C^2 - AD}}. \tag{39}$$

As it can be seen, there exist two solutions, one negative and one positive: one on the positive z-axis and one on the negative z-axis. Since only the positive one is acceptable in our case (being located in front of the camera), the coordinates of the center of the circle with respect to the $X'Y'Z'$-frame are:

$$\begin{cases} X_o' = -\dfrac{B}{A}\,Z_o' \\ Y_o' = -\dfrac{C}{A}\,Z_o' \\ Z_o' = \pm \dfrac{A\,r}{\sqrt{B^2 + C^2 - AD}}. \end{cases} \tag{40}$$

Note that the sign of the coordinate Z_o' must be selected such that the coordinate z_o in the xyz-frame would be positive.

To estimate the coordinates of the circle's center with respect to the xyz-frame (the camera frame), one must apply the following transformation:

$$\begin{bmatrix} x_o \\ y_o \\ z_o \\ 1 \end{bmatrix} = T \begin{bmatrix} X_o' \\ Y_o' \\ Z_o' \\ 1 \end{bmatrix}. \tag{41}$$

4.2 Case (II): Radius is not known

In order to solve this problem, one has to use the information from two separate images of a circle. As was discussed earlier, there exist two solutions for the orientation (norm) of a circle. Thus, one must have two images of the same circle (acquired at two distinct, but known positions) in order to determine the unique and acceptable orientation of a circle. One can use the same two images for position determination of the circle. However, the question is "how to move the camera from position (1) to position (2)?" in order to be able to solve this problem. In order to simplify the problem, it is proposed to move the camera only along its z-axis. Thus, only the z-coordinate of the circle's center would change (with respect to the camera frame). Let the initial and final coordinates of the center with respect to the camera frame be:

$$\begin{cases} x_{o1} \\ y_{o1} \\ z_{o1} \end{cases}, \quad \begin{cases} x_{o2} = x_{o1} \\ y_{o2} = y_{o1} \\ z_{o2} = z_{o1} + h. \end{cases} \tag{42}$$

Note that the value of h is known (since the length of the displacement of the camera is under control). Knowing the coefficients of the equation of the plane of a circle with respect to the camera frame, (l, m, n), one can estimate the transformation (6) as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}. \tag{43}$$

This transformation is valid for both camera positions, since the orientation of the camera does not change during relocation. Thus, one can derive:

$$\begin{cases} z_{o1} = l_3x_{o1}'' + m_3y_{o1}'' + n_3z_{o1}'' \\ z_{o2} = l_3x_{o2}'' + m_3y_{o2}'' + n_3z_{o2}''. \end{cases} \tag{44}$$

Using (44), and the first two equations of (42), the third equation of (42) would yield the following relations:

$$n_3(z_{o2}'' - z_{o1}'') = h. \tag{45}$$

Now, if one defines both z_{o1}'' and z_{o2}'' in terms of r (the unknown radius of a circle), then one can solve (45) and find r; as a result of which the problem reduces to a problem of the first case.

Let the equation of the first cone corresponding to the first camera position be

$$a_1x_1^2 + b_1y_1^2 + c_1z_1^2 + 2f_1y_1z_1 + 2g_1z_1x_1 + 2h_1x_1y_1 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1 = 0.$$

Applying the transformation (43) to the above equation, the equation of a circle with respect to the $x_1''y_1''z_1''$-frame (the camera frame in its initial position) would be obtained. As it was shown earlier, in the derived equation, the coefficients of x'' and y'' are equal, and the coefficient of the $x''y''$-term is equal to zero. Thus, if

$$A_1 \equiv a_1l_1^2 + b_1l_2^2 + c_1l_3^2 + 2f_1l_2l_3 + 2g_1l_1l_3 + 2h_1l_1l_2$$
$$B_1 \equiv a_1l_1n_1 + b_1l_2n_2 + c_1l_3n_3 + f_1l_2n_3 + f_1l_3n_2 + g_1l_3n_1 + g_1n_3l_1 + h_1l_1n_2 + h_1l_2n_1$$
$$C_1 \equiv u_1l_1 + v_1l_2 + w_1l_3$$
$$D_1 \equiv a_1m_1n_1 + b_1m_2n_2 + c_1m_3n_3 + f_1m_2n_3 + f_1m_3n_2 + g_1m_3n_1 + g_1m_1n_3 + h_1m_1n_2 + h_1m_2n_1$$
$$E_1 \equiv u_1m_1 + v_1m_2 + w_1m_3$$
$$F_1 \equiv 2(u_1n_1 + v_1n_2 + w_1n_3), \tag{46}$$

then the equation of the circle becomes (in its standard form):

$$\left[x_1'' + \left[\frac{B_1z_1''+C_1}{A_1}\right]\right]^2 + \left[y_1'' + \left[\frac{D_1z_1''+E_1}{A_1}\right]\right]^2 = \left[\frac{B_1z_1''+C_1}{A_1}\right]^2 + \left[\frac{D_1z_1''+E_1}{A_1}\right]^2 - \left[\frac{F_1z_1''+d_1}{A_1}\right]. \tag{47}$$

From equation (47), one can get the radius of a circle in terms of z_{o1}' (x_{o1}'', y_{o1}'', and z_{o1}'' are assigned as the coordinates of the center):

$$\left[\frac{B_1z_1''+C_1}{A_1}\right]^2 + \left[\frac{D_1z_1''+E_1}{A_1}\right]^2 - \left[\frac{F_1z_1''+d_1}{A_1}\right] = r^2. \tag{48}$$

Let,

$$A_1' \equiv B_1^2 + D_1^2$$
$$B_1' \equiv 2B_1C_1 + 2D_1E_1 - A_1F_1$$
$$C_1' \equiv C_1^2 + E_1^2 - A_1d_1$$
$$D_1' \equiv - A_1^2 .$$

Then (48) becomes:

$$A_1' \; z_{o1}''^2 + B_1' z_{o1}'' + (C_1' + D_1' r^2) = 0 \qquad (49)$$

from which one can estimate z_{o1}'':

$$z_{o1}'' = \frac{-B_1' \pm \sqrt{B_1'^2 - 4A_1'(C_1' + D_1' r^2)}}{2A_1'}. \qquad (50)$$

A similar equation can be derived for z_{o2}':

$$z_{o2}'' = \frac{-B_2' \pm \sqrt{B_2'^2 - 4A_2'(C_2' + D_2' r^2)}}{2A_2'}. \qquad (51)$$

Using (50) and (51), equation (45) becomes:

$$n_3 \left[\frac{-B_2'(\pm)\sqrt{B_2'^2 - 4A_2'(C_2' + D_2' r^2)}}{2A_2'} - \frac{-B_1'(\pm)\sqrt{B_1'^2 - 4A_1'(C_1' + D_1' r^2)}}{2A_1'} \right] = h . \qquad (52)$$

Equation (52), after several rearrangements and simplifications, is reduced to the following general form:

$$A''' r^4 + B''' r^2 + C''' = 0. \qquad (53)$$

Thus, the value of r can be estimated from the following equation:

$$r = \left[\frac{-B'''(\pm)\sqrt{B'''^2 - 4A'''C'''}}{2A'''} \right]^{1/2}. \qquad (54)$$

Based on physical conditions, there must be only one acceptable solution. Such physical conditions are manifested in the following constraints: the acceptable value of r has to be real and positive; r must yield positive values for z_{o1}' and z_{o2}' (equations (50) and (51)); and z_{o2}' must be greater than z_{o1}'. The above four constraints on the maximum four solutions of r are sufficient for determining the unique acceptable solution. Having estimated r, z_{o1}'' can be calculated using (50). Furthermore, from (47) one can estimate the other two coordinates of the center:

$$\begin{cases} x_{o1}'' = -\dfrac{B_1 Z_{o1}'' + C_1}{A_1} \\ y_{o1}'' = -\dfrac{D_1 Z_{o1}'' + E_1}{A_1} . \end{cases} \qquad (55)$$

These are the center coordinates with respect to the $x''y''z''$-frame. Applying the transformation (43), one can get the center coordinates with respect to the xyz-frame (the camera frame).

5. Application of the Developed Method to 3D Quadratic Surfaces

There exist two possible ways to extend the developed mathematical method to other surfaces or features. On the one hand, the problem can be defined as: given a quadratic surface (ellipsoid, paraboloid, hyperboloid, and cylindroid), find the orientation of the plane that intersects the given surface and generates a circular curve. It can be shown that applying the general transformation (22), and following the same procedure formulated in section 3, a set of unique solutions can be obtained for each of the above surfaces. However, since this problem and its solution are not applicable to the 3D-object recognition method under development, though otherwise having mathematical merit, the details of the solution are not presented here.

On the other hand, the problem can be defined as: given a 3D feature (as opposed to a 2D planar feature that was addressed in the preceding sections), find its location with respect to the camera frame. One of the 3D features that might be used in 3D-model-based vision is a spherical feature [7]. Thus, the problem can be defined more specifically as: given the radius of a sphere, its perspective projection, and the effective focal length of a camera, determine its location with respect to the camera frame. This problem is schematically shown in Figure 3.

Let

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 . \qquad (56)$$

be the equation of a quadratic surface. Furthermore, let the equations of the straight lines through a point $P(\alpha,\beta,\gamma)$, whose direction-ratios are l_1, m_1, n_1 and are tangent to the above quadratic surface, be

$$\frac{x-\alpha}{l_1} = \frac{y-\beta}{m_1} = \frac{z-\gamma}{n_1}. \qquad (57)$$

These tangent lines generate a cone that envelopes the quadratic surface. It has been proven [9] that the equation of such an enveloping surface (an enveloping cone) is:

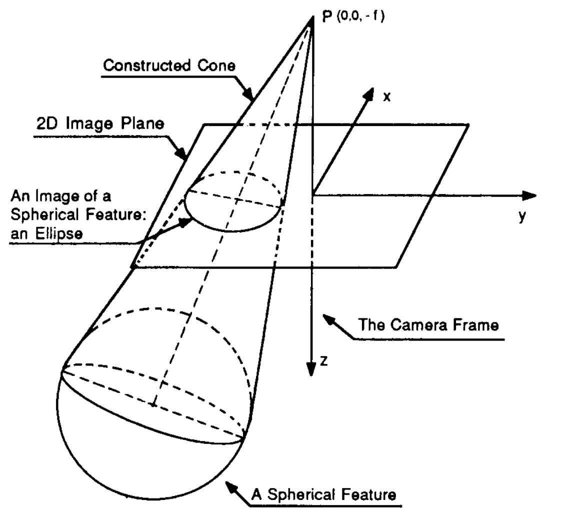

Figure 3. Schematic representation of the problem for a spherical feature.

$$4F(\alpha,\beta,\gamma)\; F(x,y,z) = \left[(x-\alpha)\frac{\partial F}{\partial \alpha} + (y-\beta)\frac{\partial F}{\partial \beta} + (z-\gamma)\frac{\partial F}{\partial \gamma} + 2F(\alpha,\beta,\gamma) \right]^2 . \qquad (58)$$

For the special case under consideration, a sphere, the general equation of the surface is defined (with respect to the camera frame) as follows:

$$F(x,y,z) \equiv (x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2 - r^2 = 0 . \qquad (59)$$

Furthermore, the coordinates of the point P (as the origin of the camera frame) with respect to the camera frame would be (0,0,0). Then, the equation of the enveloping cone (with respect to the camera frame) becomes:

$$(x_o^2 + y_o^2 + z_o^2 - r^2)\left[(x-x_o)^2 + (y-y_o) + (z-z_o)^2 - r^2\right] = \left[-x_o x - y_o y - z_o z + (x_o^2 + y_o^2 + z_o^2 - r^2)\right]^2 . \qquad (60)$$

Now, if the camera frame is rotated such that the new z'-axis passes through the center of the sphere, then the sphere's center coordinates would be $(0,0,z_o')$, and the equation of the enveloping cone with respect to the new $x'y'z'$-frame would be (note that $z_o' > r$):

$$x'^2 + y'^2 - \left[\frac{r^2}{z_o'^2 - r^2} \right] z'^2 = 0 . \qquad (61)$$

This is the equation of a cone in its central form. Thus, the above-mentioned rotation of the camera frame is the same as the rotational transformation (15). Furthermore, two of the three coefficients in equation (61) are equal, which corresponds to case IV in section 3.2. As was noted earlier, this case results in a right circular cone; That is, the principal axis of the cone is perpendicular to its circular base. From a geometrical point of view, this conclusion was expected.

Thus, the analytical solution for the problem would be as follows:

1. Given the effective focal length and the parameters of the perspective projection of a spherical feature - an ellipse - with respect to the camera frame (the xyz-frame), the equation of a cone can be obtained using equation (2).

2. Applying the rotational transformation (15), the equation of the cone in its central form (17) is obtained with respect to the rotated frame (the $x'y'z'$-frame).

3. Normalizing the first two coefficients of equation (17) to 1, and equating the third coefficients of equations (17) and (61), the unknown value z_o' is derived:

$$z_o' = \pm \left[\frac{1 + \lambda_3/\lambda_1}{\lambda_3/\lambda_1} \right]^{1/2} r. \qquad (62)$$

The positive solution, being in front of the camera, is the unique acceptable solution for the z_o' coordinate. Thus, the center coordinates with respect to the rotated frame would be $(0,0,z_o')$.

4. The 3D coordinates of the center of the spherical feature with respect to the camera frame is determined by applying the rotational transformation (15) to the estimated 3D coordinates in step 3.

6. Experimental Results

Accurate estimation of the 3D-location of a circular feature from an input grey-level image requires a general analytical solution method, as well as other processes to compensate for various types of distortion. In a real process, as opposed to a simulated process, various sources of noise affect the input image and thus distort it. The experimental results in this section report the total process of accurate estimation of the 3D-location of a circular feature, which in part, involves the general 3D-analytical-solution method derived in this paper (without using any simplifying assumptions).

The details of the various steps required for this purpose, which have already been addressed in other published papers, are not presented here. However, a brief review of these steps is presented below:

(1) *Camera calibration:* The camera is calibrated by applying the mono-view non-coplanar points technique [10], as a result of which, the 3D location of the camera frame with respect to a pre-defined world frame of reference is estimated. Furthermore, the effective focal length of the camera, the radial distortion factor of the camera's lens, and the uncertainty scale factor for the x-axis (due to the timing mismatches which occur between image-acquisition hardware and camera scanning hardware) are also obtained.

(2) *Sub-pixel edge detection:* After an image of a circular feature is acquired, a new sub-pixel edge detector is applied [11]. This edge operator is based on the sample-moment-preserving transform (SMPT) and assumes a circular-arc geometry for the boundary inside the detecting area. The result of the edge detector is a set of sub-pixel edge-points data. The sub-pixel edge detector compensates for quantization error and estimates the boundary of a circular feature more reliably.

(3) *Coordinate transformation:* Computer-image coordinates are expressed in terms of pixel units. To define the edge-points in terms of absolute length units (in mm), and also to compensate for timing mismatches, a set of transformations are applied [10]. This is implemented by using the uncertainty factor estimated in step (1) and some technical specifications of the camera's CCD chip and the digitizer board.

(4) *Lens-radial-distortion compensation:* The estimated lens-radial-distortion factor in step (1) is applied to all edge-points to compensate for the lens radial distortion [10].

(5) *Elliptical-shape-parameters estimation:* An interpolation technique based on the optimization of an error function is applied to accurately estimate the five basic parameters of an ellipse - the perspective projection of a circle onto the image plane [12].

(6) *Circular-feature 3D-orientation estimation:* Using the estimated effective focal length in step (1) and the estimated values for the five basic parameters of an ellipse in step (5), and applying the analytical method developed in section 3, the orientation of the circular feature with respect to the camera frame is estimated.

(7) *Circular-feature 3D-position estimation:* Using the estimated orientation of a circular feature in step (6) and its known radius, and applying the analytical method developed in section 4, the 3D position of the feature is estimated with respect to the camera frame. Applying the transformation from the camera frame to the world frame of reference, obtained in step (1), to the estimated 3D position of the circle yields the 3D position with respect to the world-reference frame.

For experimentation on the total process, six co-planar circles arranged into two columns on the left and right sides of the plane were used. The plane of the circles, in an inclined orientation with respect to the camera image plane, was positioned such that the circles covered the entire field of view. These conditions provided the most general camera-circular-feature configuration. Furthermore, in order to obtain a sharp image of all circles, this plane was located within the approximated existing depth of field of the camera [13].

The application of the above seven-step procedure to the six coplanar circles, resulted in two sets of data, tabulated in Tables 1 and 2. Through camera calibration, the orientation angles of the normal to the circles' plane was estimated. These are referred to as "Reference Angles" in Table 1. The estimated orientation angles of each circle's norm are also presented in this table. Note that since the circles were coplanar, they must have the same orientation angles. The average orientation angle is defined as the mean value of orientation angles of the six circles, while the average deviation is defined as the absolute value of the difference between a reference angle and an average angle. The average deviations for the three orientation angles were determined as, 0.80, 0.34, and 0.41 degree respectively. As it can be seen, the results show a small amount of error which indicates good performance of the total process.

In Table 2, the results of the position-estimation process are presented. The coordinates, estimated with respect to the world reference frame, are given under the column "Estimated". The exact 3D coordinates of the circles' centers are known a priori and are given in Table 2 under the column "Reference". The differences between the reference and the estimated coordinates of all the circles are calculated, and the means of these values are given under "Average Deviations" The results can be better appreciated when the size of the field of view (275 mm by 200 mm) and the focused distance (864 mm) are taken into consideration. The 1.28 mm average error for the depth estimation in an approximately 864 mm focused distance is less than 1.5 parts in 1000 average accuracy. As a whole, both sets of results show the validity of the total process involved in the 3D-location estimation in general, and the applicability of the analytical method developed in this paper in particular.

Table 1. Estimated orientation angles of the surface normals of a set of circular features.

Angles	α (degree)	β (degree)	γ (degree)
Reference Angles	89.72	76.73	13.27
Circle # 1	88.73	76.06	13.99
Circle # 2	89.61	74.99	15.01
Circle # 3	88.75	76.54	13.52
Circle # 4	89.87	76.39	13.61
Circle # 5	89.11	76.43	13.60
Circle # 6	87.46	77.93	12.34
Average Angles	88.92	76.39	13.68
Average Deviations	0.80	0.34	0.41

Note: α, β, γ are the angles which the surface normal of a circle makes with the x, y, z axes of the camera frame respectively.

Table 2. Estimated positions of a set of circular features.

Coordinates	x (mm)		y (mm)		z (mm)	
Circle No.	Reference	Estimated	Reference	Estimated	Reference	Estimated
# 1	0.00	0.38	0.00	0.63	10.00	8.05
# 2	185.00	184.78	0.00	- 0.03	10.00	11.01
# 3	0.00	0.24	37.00	37.27	10.00	9.31
# 4	185.00	184.64	37.00	37.43	10.00	9.15
# 5	0.00	0.26	74.00	74.34	10.00	8.92
# 6	0.00	0.46	148.00	148.60	10.00	7.92
Average Deviations	0.32		0.38		1.28	

Note: All the coordinates are with respect to the world reference frame.

Conclusions

Accurate 3D-location estimation of a circular marker feature is a major problem to be solved for the 3D-object-recognition method under development at the University of Toronto. In this paper, the problem of orientation estimation of a circular feature was addressed first. An analytical formulation of the problem and an iterative solution, based on 3D analytical geometry, were presented. Subsequently, a closed-form analytical solution was derived by reducing the general equation of a cone to its central form, in order to develop a more efficient method. For position estimation of a circular feature, two closed-form solution methods corresponding to two possible cases, whether the radius of a feature is known or it is not, were derived. General extension of the method to quadratic surfaces was addressed and a unique analytical solution for a 3D spherical feature was obtained. In order to verify the developed method, it was applied to set of circular features. The camera was calibrated prior to the application of the method. Also, in order to obtain accurate estimates of the parameters of the imaged circle - an ellipse - a sequential compensation procedure was applied to the input grey-level image. The experimental results obtained show the validity of the method developed.

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