A Theory of Well-Connected Relations

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ABSTRACT

The simple concept of a well-connected relation (WCR) is defined and then developed into a theory for studying the structure of binary relations. Several interesting partitions of a binary relation are identified. It is shown that binary relations can be separated into families which form a hierarchy. Finally, concepts of universal algebra are extended to the family of "prime" relations. One application for the theory is in the logical design of data bases.

INTRODUCTION

Data in an information system have been studied from several points of view [2, 4, 9, 10, 11]. With the growing interest in data base systems, the possibility for structuring the data in various ways has been explored by several authors. For example, Delobel [9] defines "elementary functional relations" and proposes a theory based on this building block; Zaniolo [10] defines "atomic relations" as a basic building block for data; Furtado and Kerschberg [11] have "quotient relations." Further, many authors have noted and studied specific data structures, such as "functional dependencies" [4], "multivalued dependencies" [5], "contextual dependencies" [6], "mutual dependencies" [12] (which happen to be same as contextual dependencies), and "hierarchical dependencies" [13].

In this paper we define a new building block, "well-connected relations," and propose a theory to study binary relations. Elsewhere [6] we have applied this theory to the study of dependency structures of data. We have also studied

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a manipulation language for these building blocks [8]. An overview of these applications is given as part of the conclusion.

This paper is divided into three parts. Part I introduces the basic concept of a well-connected relation (WCR) and a few of its properties which are useful later. Part II uses a WCR as the basic building block of any binary relation. Several interesting partitions of a binary relation are identified, as a result of which they can be separated into families that form a hierarchy. Part III studies one such family from universal algebraic point of view. It is to be noted that the Part III just introduces the broad concept of an algebra of WCRs without going into the details of the operations. The operations have been studied elsewhere [8].

PART I

The concept of a binary relation may be found in the appropriate books of mathematics (for example [1]). In this part, we define and study a well-connected relation, which is nothing but a cross product between two sets. We also review some of the set theoretic operations for the sake of completeness. The concepts of contraction and expansion are introduced because they are useful [8]. Some of the theorems proved here will be used in the subsequent parts of this paper.

DEFINITION 1. A *binary relation* $R[A,B]$ on the sets A and B is a mapping from set A to set B such that every element of A is mapped to at least one element of B.

The binary relation is *partial* if some elements of A are not mapped to any element of B.

DEFINITION 2. A *well-connected relation* (WCR) is a binary relation W on two sets A and B such that

$$(\forall x)(x \in A)(\forall y)(y \in B)(x\, W y). \tag{1}$$

The sets A and B are called the *constituents* of the WCR.

NOTE 1.

(i) A binary relation is $R[A,B]$, while a WCR is $W[A,B]$. We also use $S_R[AB]$ and $S_R[A]$, which mean the following:

$$S_R[AB] = \{(a,b) : (a \in A), (b \in B), (a R b)\}$$

$$= S_R, \tag{2}$$

$$S_R[A] = \{(b) : (b \in B) \text{ and } (\exists a)(a \in A)(a R b)\} \tag{2a}$$

$$= B = \text{Image set of } A \text{ in } R[A,B]. \tag{3}$$

By definition,

$$S_R[AB]=\varnothing \quad \text{iff} \quad A=\varnothing \text{ or } B=\varnothing.$$

(ii) In this paper there is no need to consider partial binary relations. Also the mapping from set A to set B is always taken to be onto.

Set operations can be defined for binary relations.

(i) The *union* of two relations R and R_1 is

$$R_2[A_2, B_2] = R[A, B] \cup R_1[A_1, B_1],$$

where

$$S_{R_2}[A_2B_2] = \{(a,b): a \in (A_2 = A \cup A_1),\ b \in (B_2 = B \cup B_1) \text{ and } (a\,R\,b) \text{ or } (a\,R_1\,b)\}. \tag{4}$$

(ii) The *intersection* of two relations R and R_1 is

$$R_2[A_2, B_2] = R[A, B] \cap R_1[A_1, B_1],$$

where

$$S_{R_2}[A_2B_2] = \{(a,b): a \in (A_2 \subseteq A \cap A_1),\ b \in (B_2 \subseteq B \cap B_1), (a\,R\,b) \text{ and } (a\,R_1\,b)\}. \tag{5}$$

The relations R and R_1 are disjoint if S_{R_2} is null.

(iii) The *difference* of two relations R and R_1 is

$$R_2[A_2, B_2] = R[A, B] - R_1[A_1, B_1],$$

where

$$S_{R_2}[A_2B_2] = \{(a,b): a \in (A_2 \subseteq A),\ b \in (B_2 \subseteq B), (a\,R\,b) \text{ and } (a\not R_1\,b)\}. \tag{6}$$

From the above definition it is clear that

$$S_{R_2}[B_2] \subseteq S_R[B] \quad \text{and} \quad S_{R_2}[A_2] \subseteq S_R[A].$$

(iv) *Containment*: A relation R is contained in another relation R_1, i.e.,

$$R[A, B] \subseteq R_1[A_1, B_1],$$

if

$$A \subseteq A_1, \quad B \subseteq B_1 \quad \text{and} \quad S_R[AB] \subseteq S_{R_1}[A_1B_1]. \tag{7}$$

It is a proper containment, i.e.,

$$R[A,B] \subset R_1[A_1,B_1],$$

if

$$A \subseteq A_1, \quad B \subseteq B_1 \quad \text{and} \quad S_R[AB] \subset S_{R_1}[A_1B_1]. \tag{8}$$

It is equality, i.e.,

$$R[A,B] = R_1[A_1,B_1],$$

if

$$A = A_1, \quad B = B_1 \quad \text{and} \quad S_R[AB] = S_{R_1}[A_1B_1]. \tag{8a}$$

(v) The *complement* of $R[A,B]$ with respect to $R_1[A_1,B_1]$ is defined if $R[A,B] \subseteq R_1[A_1,B_1]$. It is

$$\bar{R}[A,B] = R_2[A_2,B_2],$$

where

$$S_{R_2}[A_2B_2] = \{(a,b) : a \in (A_2 \subseteq A_1),\ B \in (B_2 \subseteq B_1),$$
$$(A\, R_1\, b) \text{ and } (A \not R b)\}. \tag{9}$$

THEOREM 1. *The intersection of two WCRs is a WCR.*

Proof. Let $W[A,B]$ and $W_1[A_1,B_1]$ be the two WCRs, and let $R_2[A_2,B_2] = W \cap W_1$.

Assume R_2 is not a WCR, i.e.,

$$(\exists x)(x \in A_2)(\exists y)(y \in B_2)(x \not R_2 y) \qquad \text{[from (1)]}$$

But $A_2 \subseteq A \cap A_1$ and $B_2 \subseteq B \cap B_1$ [from (5)].

$$\therefore \qquad (\exists x)(x \in A \cap A_1)(\exists y)(y \in B \cap B_1)(x \not R_2 y). \tag{10}$$

But

$$(\forall x)(x \in A)(\forall y)(y \in B)(x\, W y)$$

and

$$(\forall x)(x \in A_1)(\forall y)(y \in B_1)(x\, W_1 y) \qquad \text{[from (1)]}.$$

Hence,

$$(\forall x)(x \in A \cap A_1)(\forall y)(y \in B \cap B_1)\ (x\, W y) \text{ and } (x\, W_1 y).$$

Hence,

$$(\forall x)(x \in A \cap A_1)(\forall y)(y \in B \cap B_1)(x\, R_2 y) \qquad \text{[from (5)]}. \qquad (11)$$

Here (11) contradicts (10). Hence the assumption is wrong, i.e., R_2 is a WCR.

DEFINITION 3. The *contraction* of a binary relation $R[A,B]$ on the sets A_1 and B_1, where $A_1 \subseteq A$ and $B_1 \subseteq B$, is the binary relation $R_1[A_1,B_1]$ such that

$$S_{R_1}[A_1 B_1] = \{(a,b) : a \in A_1,\ b \in B_1,\ (a R b)\} \qquad (12)$$

It is a *minimal contraction* if $A_1 = A$ or $B_1 = B$.
It is a *null contraction* if $A_1 = A$ and $B_1 = B$.

DEFINITION 4. The *expansion* of a binary relation $R[A,B]$ within another binary relation $R_1[A_1,B_1]$ on the sets A_2 and B_2 is defined if $R_1[A_1,B_1]$ contains $R[A,B]$ and $A \subseteq A_2 \subseteq A_1$ and $B \subseteq B_2 \subseteq B_1$. It is a binary relation $R_2[A_2,B_2]$ such that

$$S_{R_2}[A_2 B_2] = \{(a,b) : a \in A_2,\ b \in B_2,\ (a R_1 b)\} \qquad (13)$$

It is a *minimal expansion* if $A_2 = A$ or $B_2 = B$.
It is a *null expansion* if $A_2 = A$ and $B_2 = B$.

THEOREM 2. *Any contraction of a WCR is a WCR.*

Proof. Let $W[A,B]$ be a WCR, and let $M_1[A_1,B_1]$ be a contraction of W. Then

$$A_1 \subseteq A \quad \text{and} \quad B_1 \subseteq B \qquad \text{[from Definition 3]}.$$

Now,

$$(\forall x)(x \in A)(\forall y)(y \in B)(x\,W y) \qquad \text{[from (1)]}.$$

Substituting A_1 and B_1 for A and B, we can say

$$(\forall x)(x \in A_1)(\forall y)(y \in B_1)(x\,W y) \quad \Rightarrow \quad x\,M_1 y \qquad \text{[from (12) and (2)]}.$$

Hence,

$$(\forall x)(x \in A_1)(\forall y)(y \in B_1)(x\,M_1 y) \quad \Rightarrow \quad M_1 \text{ is a WCR} \qquad \text{[from (1)]}.$$

COROLLARY 1. *The complement of a minimal contraction of a WCR with respect to the same WCR is a WCR.*

Proof. The proof is got by showing that the complement itself is a minimal contraction of the given WCR.

THEOREM 3. *Any minimal expansion of the intersection of two WCRs within the union of the same two WCRs is a WCR.*

Proof. Let

$$R_3[A_3, B_3] = W_1[A_1, B_1] \cup W_2[A_2, B_2], \tag{14}$$

$$W_4[A_4, B_4] = W_1[A_1, B_1] \cap W_2[A_2, B_2] \qquad \text{[from Theorem 1]} \tag{15}$$

and

$$M_5[A_5, B_5] = \text{a minimal expansion of } W_4[A_4, B_4] \text{ within } R_3[A_3, B_3]. \tag{16}$$

From the definition of minimal expansion, let

$$A_5 = A_4 \quad \text{and} \quad B_4 \subseteq B_5 \subseteq B_3. \tag{17}$$

We have to show that $M_5[A_5, B_5]$ is a WCR, i.e., $(\forall x)(x \in A_5](\forall y)(y \in B_5)(x\,M_5 y)$. We have

$$(\forall x)(x \in A_1 \cap A_2)(\forall y)(y \in B_1)(x W_1 y)$$

and

$$(\forall x)(x \in A_1 \cap A_2)(\forall y)(y \in B_2)(x W_2 y)$$

from (1) and Theorem 2. From the above two expressions, with (4) and (14),

$$(\forall x)(x \in A_1 \cap A_2)(\forall y)(y \in B_1 \cup B_2)(x R_3 y).$$

This is the definition of a WCR, i.e., the minimal expansion of the intersection of W_1 and W_2 within R_3 is a WCR if $B_5 = B_3$. Again, $A_5 = A_4 \subseteq A_1 \cap A_2$ [from (5), (15) and (17)], and $B_5 \subseteq (B_3 = B_1 \cup B_2)$ [from (4), (14) and (17)].

Applying Theorem 2, we can write

$$(\forall x)(x \in A_5)(\forall y)(y \in B_5)(x M_5 y) \text{ [from (13) and (16)],}$$

i.e., M_5 is a WCR.

THEOREM 4. *Two WCRs, $W_1[A_1, B_1]$ and $W_2[A_2, B_2]$, are disjoint if and only if A_1 is disjoint from A_2 or B_1 is disjoint from B_2.*

Proof. Let $A_1 \cap A_2 = \varnothing$ and $W_1 \cap W_2 = W_2 = W_3$. Then

$$S_{W_3}[A_3 B_3] = \{(a,b) : a \in (A_3 \subseteq A_1 \cap A_2),\ b \in (B_3 \subseteq B_1 \cap B_2),$$
$$(a W_1 b) \text{ and } (a W_2 b)\} \qquad \text{[from (5)].}$$

But $A_1 \cap A_2 = \varnothing$. Hence $A_3 = \varnothing$, i.e., $S_{W_3}[A_3 B_3] = \varnothing$ [from (2a)], i.e., W_1 and W_2 are disjoint.

Again, let

$$W_1 \cap W_2 = W_3, \qquad \text{where} \quad S_{W_3}[A_3 B_3] = \varnothing.$$

Then

$$S_{W_3}[A_3 B_3] = \{(a,b) : a \in (A_3 \subseteq A_1 \subseteq A_2),\ b \in (B_3 \subseteq B_1 \cap B_2),$$
$$(a W_1 b) \text{ and } (a W_2 b)\} \qquad \text{[from (5)].} \tag{18}$$

We have

$$(\forall x)(x \in A_1)(\forall y)(y \in B_1)(x W_1 y),$$

$$(\forall x)(x \in A_2)(\forall y)(y \in B_2)(x W_2 y)$$

from (1). From the above two expressions, we can write

$$(\forall x)(x \in A_1 \cap A_2)(\forall y)(y \in B_1 \cap B_2)\ (x W_1 y) \text{ and } (x W_2 y). \tag{19}$$

From (18) and (19), we can write

$$S_{W_3}[A_3B_3]=\{(a,b): a\in(A_3=A_1\cap A_2),\ b\in(B_3=B_1\cap B_2),$$
$$(a\,W_1\,b) \text{ and } (a\,W_2\,b)\}.$$

But $S_{W_3}[A_3B_3]=\varnothing$. Hence $A_3=A_1\cap A_2=\varnothing$ or $B_3=B_1\cap B_2=\varnothing$ [from (2a)].

NOTE 2. Consider a binary relation $R[A,B]$ where

$$S_R[AB]=\varnothing.$$

Then R by definition is a WCR.

THEOREM 5. *Two WCRs are equal if and only if their constitutents are equal.*

Proof. The proof is trivial and follows from the definitions of a WCR and the equality of two WCRs, i.e., from (1) and (8a).

PART II

In this part we establish a hierarchy of the families of binary relations (Theorem 16). The families of binary relations are based on different ways the binary relations can be partitioned. Some of these partitions are unique for the binary relation, and this fact is brought out by appropriate theorems. Also, algorithms are given to find these unique partitions. Finally, this section also contains theorems to establish different parts of the hierarchy, which is summed up neatly in Theorem 16.

NOTE 3. We use the following terminology:

(i) A set A can be expressed as

$$A=\coprod_{i=1}^{n} A_i=A_1\cup A_2\cup\cdots\cup A_n$$
$$=\pi(A)=\text{partition of } A,$$

where $A_i\cap A_j=\varnothing$ for $1\leqslant i,j\leqslant n$ and $i\neq j$.

(ii) A set A can be expressed as

$$A=\bigcup_{i=1}^{n} A_i=A_1\cup A_2\cdots\cup A_n$$
$$=C(A)=\text{cover of } A,$$

where $A_i \cap A_j \supseteq \varnothing$ for $1 \leqslant i, j \leqslant n$.

(iii) A relation $R[A,B]$ can be expressed as

$$\begin{aligned} R[A,B] &= \prod_{i=1}^{n} R_i[A_i,B_i] \\ &= R_1[A_1,B_1] \cup R_2[A_2,B_2] \cup \cdots \cup R_n[A_n,B_n] \\ &= \pi(R) = \text{partition of } R, \end{aligned}$$

where $R_i[A_i,B_i] \cap R_j[A_j,B_j] = \varnothing$ for $i \neq j$, $1 \leqslant i,j \leqslant n$, and

$$A = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{n} B_i.$$

(iv) A relation $R[A,B]$ can be expressed as

$$\begin{aligned} R[A,B] &= \bigcup_{i=1}^{n} R_i[A_i,B_i] \\ &= R_1[A_1,B_1] \cup R_2[A_2,B_2] \cup \cdots \cup R_n[A_n,B_n] \\ &= C(R) = \text{cover of } R, \end{aligned}$$

where $R_i[A_i,B_i] \cap R_j[A_j,B_j] \geqslant \varnothing$ for $1 \leqslant i,j \leqslant n$, and

$$A = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{n} B_i.$$

DEFINITION 5. Two partitions, $\pi_1(R)$ and $\pi_2(R)$, of a relation $R[A,B]$ are *equal* if

$$\pi_1(R) = \prod_{i=1}^{n} R_{1i}[A_{1i},B_{1i}],$$

$$\pi_2(R) = \prod_{i=1}^{n} R_{2i}[A_{2i},B_{2i}],$$

and

$$(\forall i)(\exists j)(R_{1i} = R_{2j})(1 \leqslant i,j \leqslant n).$$

THEOREM 6. *Any binary relation is a partition of WCRs.*

Proof. A trivial partition of WCRs of any binary relation is one in which each tuple of the binary relation is considered to be a WCR.

DEFINITION 6. A partition

$$R[A,B]=\prod_{i=1}^{n} W_i[A_i,B_i]$$

of a binary relation $R[A,B]$ is a *canonical partition* if

(i) $W_i[A_i,B_i]$ is a WCR for $1 \leqslant i \leqslant n$,
(ii) B_i is a set with a single element for $1 \leqslant i \leqslant n$,
(iii) $B_i \neq B_j$ for $i \neq j$ and $1 \leqslant i,j \leqslant n$.

THEOREM 7. *Any binary relation has one and only one canonical partition.*

Proof. Let $R[A,B]$ be a binary relation. Let

$$B=\{b_1,b_2,\ldots,b_n\}, \tag{20}$$

where $b_i \neq b_j$ for $i \neq j$ and $i \leqslant i,j \leqslant n$;

$$B_i=\{b_i\} \text{ for } i \leqslant i \leqslant n; \tag{21}$$

$$A_i=\{a:(a \in A),(b_i \in B),(aRb_i)\} \qquad \text{for} \quad i \leqslant i \leqslant n. \tag{22}$$

Now, for $1 \leqslant i \leqslant n$,

$$A_i=\varnothing \qquad \text{[from Definition 1 and Note 1(ii)]}, \tag{23}$$

$$R_i[A_i,B_i]=\varnothing \qquad \text{[from (21), (22) and (23)]}$$

$$=W_i[A_i,B_i] \qquad \text{[from (1), (21) and (22)]}. \tag{24}$$

We have to show that

$$R[A,B]=\prod_{i=1}^{n} W_i[A_i,B_i]=\pi_c(R). \tag{25}$$

It then follows that $\pi_c(R)$ is canonical [from Definition 6 and (20), (21) and (24)]. Hence a canonical partition exists for $R[A,B]$.

To show (25), we have to show:

(i) $A=\bigcup_{i=1}^{n} A_i$.

(ii) $W_i[A_i, B_i] \cap W_j[A_j, B_j] = \varnothing$ for $1 \leqslant i, j \leqslant n$ and $i \neq j$.
(iii) $R[A,B] = W_1[A_1, B_1] \cup W_2[A_2, B_2] \cup \cdots \cup W_n[A_n, B_n] = C_W(R)$.

(i) Assume

$$A \neq \bigcup_{i=1}^{n} A_i.$$

But $A \not\subset \bigcup_{i=1}^{n} A_i$ ($\because$ each $A_i \subseteq A$) [from (22)].

$$\therefore \qquad A \supset \bigcup_{i=1}^{n} A_i.$$

$$\therefore \qquad (\exists a)\ (a \in A) \text{ and } \left(a \notin \bigcup_{i=1}^{n} A_i\right).$$

$\therefore$ $\quad (\exists b_j)\ (b_j \in B)$ and $(a R b_j)$ [from Definition 1 and Note 1 (ii)].

$\therefore$ $\quad a \in A_j$ [from (22)].

Hence,

$$a \in \bigcup_{i=1}^{n} A_i$$

—a contradiction. Hence,

$$A = \bigcup_{i=1}^{n} A_i.$$

(ii) We have

$B_i \cap B_j = \varnothing$ for $i \neq j$ and $1 \leqslant i, j \leqslant n$ [from (20) and (21)].

$\therefore$ $\quad W_i[A_i, B_i] \cap W_j[A_j, B_j] = \varnothing$ (from Theorem 4).

iii) Assume

$$R[A,B] \neq C_W(R).$$

$$\therefore \qquad (\exists(a,b))((a,b)\in S_R)\big((a,b)\notin\{S_{W_1}\cup S_{W_2}\cup\cdots\cup S_{W_n}\}\big).$$

$$\therefore \qquad b\in B.$$

Let

$$b=b_j \qquad \text{where} \quad 1\leqslant j\leqslant n.$$

$$\therefore \qquad (a,b)\in S_{W_j},\ \text{where } 1\leqslant j\leqslant n, \qquad \text{[from (2), (22) and (24)]}.$$

$$\therefore \qquad (a,b)\in\{S_{W_1}\cup S_{W_2}\cup\cdots\cup S_{W_n}\}$$

—a contradiction.
Hence

$$R[A,B]=C_W(R).$$

We have to show the converse, i.e., $\pi_c(R)$ is the only canonical partition of $R[A,B]$. Assume, there is another canonical partition $\pi(R)$ and

$$\pi_c(R)\neq\pi(R)$$

Let

$$\pi(R)=\prod_{i=1}^{m}\omega_i[C_i,D_i]$$

From (25),

$$\pi_c(R)=\prod_{i=1}^{n}W_i[A_i,B_i].$$

Now, for $1\leqslant i\leqslant m$, from Definition 6,

$$D_i=\{d_i\}=\text{a set with a single element} \tag{26}$$

and

$$B = \bigcup_{i=1}^{m} D_i \qquad \text{[from Note 3(iii)].} \tag{27}$$

But

$$B = \bigcup_{i=1}^{n} B_i \qquad \text{[from Note 3(iii)]} \tag{28}$$

and

$$\left.\begin{array}{ll} D_i \neq D_j & \text{for } i \neq j, \\ B_i \neq B_j & \text{for } i \neq j \end{array}\right\} \qquad \text{(from Definition 6).} \tag{28a}$$

$\therefore$

$$m = n,$$

and

$$(\exists i)(\exists j)(1 \leqslant i,j \leqslant n)(D_i = B_j), \tag{29}$$

from (21), (26), (27), (28) and (28a).

$\therefore$

$$\pi(R) = \prod_{i=1}^{n} W_i[C_i, D_i] \neq \pi_c(R).$$

$\therefore$

$$\big(\exists \omega_p[C_p, D_p]\big)\big(\omega_p[C_p, D_p] \in \pi(R)\big), \tag{30}$$

$$(\forall j)(1 \leqslant j \leqslant n)\big(\omega_p[C_p, D_p] \neq W_j[A_j, B_j]\big).$$

But

$$D_p \in \bigcup_{i=1}^{n} D_i \qquad \text{[from Note 3(iii)].}$$

$\therefore$

$$(\exists j)(1 \leqslant j \leqslant n)(D_p = B_j) \qquad \text{[from (29)].} \tag{31}$$

$\therefore$

$$\omega_p[C_p, D_p] \subseteq W_j[A_j, B_j] \qquad \text{[from (22) and (24)].}$$

But $\omega_p[C_p, D_p] \neq W_j[A_j, B_j]$ [from (30)].

$\therefore$

$$\omega_p[C_p, D_p] \subset W_j[A_j, B_j].$$

$\therefore$

$$\big(\exists (a,b)(a,b) \in S_{W_j}\big)\big((a,b) \notin S_{\omega_p}\big). \tag{32}$$

But $(a,b)\in S_R$ ($\because$ $(a,b)\in S_{W_j}$ and $W_j[A_j,B_j]\in\pi_c(R)$).

$$\therefore \qquad (a,b)\in S_{\omega_q}, \qquad \text{where} \quad \omega_q[C_q,D_q]\in\pi(R).$$

$$\therefore \qquad b\in B_j \quad \text{and} \quad b\in D_q.$$

But B_j and D_q are both sets with a single element.

$$\therefore \qquad B_j=D_q.$$

But $B_j=D_p$ [from (31)].

$$\therefore \qquad D_p=D_q.$$

$$\therefore \qquad \omega_p[C_p,D_p]=\omega_q[C_q,D_q] \qquad [\because\ \pi(R) \text{ is a canonical partition}].$$

$\therefore$ $(a,b)\in S_{\omega_p}$ and $S_{\omega_p}[C_p D_p]$corresponds to $W_j[A_j,B_j]$ [from (32)].

This is a contradiction to (30). Hence,

$$\pi_c(R)=\pi(R)=\text{the only canonical partition of } R[A,B].$$

NOTE 4. In a binary relation $R[A,B]$, let

$$A=\{a_1,a_2,\ldots,a_n\},$$

where $a_i\neq a_j$ for $i\neq j$ and $1\leqslant i,j\leqslant n$. Then

$$S_R[a_i]=\text{Image set of } \{a_i\} \text{ in relation } R[A,B]$$
$$=\{b:(b\in B),(a_i R b)\}$$

If $A_i\subseteq A$, then

$$S_R[A_i]=\text{Image set of } A_i \text{ in relation } R[A,B]$$
$$=\{b:(b\in B)(a\in A_i)(aRb)\}.$$

Algorithm 1. *To find the canonical partition of a binary relation* $R[A,B]$.

1. *Find set* $B=\{b_1,b_2,\ldots,b_n\}$, *where* $b_i\neq b_j$ *for* $i\neq j$ *and* $1\leqslant i,j\leqslant n$.
2. *Find image sets* $S_R[b_i]$ *for* $i\leqslant i\leqslant n$.

3. *Let* $\{b_i\} = B_i$ *and* $S_R[b_i] = A_i$, *and form WCRs* $W_i[A_i, B_i]$ *for* $1 \leqslant i \leqslant n$.
4. *Then*

$$\pi_c(R) = \text{the canonical partition of } R[A,B]$$

$$= \prod_{i=1}^{n} W_i[A_i, B_i] \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Proof. The proof is quite straightforward when we note the following:

(i) The set B here is the same as in (20).
(ii) The sets B_i are the same as in (21).
(iii) The image sets $S_R[b_i]$ are the same as the sets A_i in (22).
(iv) The WCRs $W_i[A_i, B_i]$ are the same as in (24).

Hence from (25) it follows that $\pi_c(R)$ is the canonical partition of $R[A,B]$.

DEFINITION 7. A partition of a binary relation $R[A,B]$ is a *strong partition* if

$$R[A,B] = \prod_{i=1}^{n} R_i[A_i, B_i] = \pi_s(R), \qquad \text{where} \quad n \geqslant 1,$$

and

$$A = \prod_{i=1}^{n} A_i \qquad B = \prod_{i=1}^{n} B_i$$

$$= \pi_I(A), \qquad = \pi_I(B).$$

Here $R_i[A_i, B_i]$ are called the *blocks* of $\pi_s(R)$; $\pi_I(A)$ and $\pi_I(B)$ are called the *image partitions* of $\pi_s(R)$; and $R[A,B]$ is called a *strong relation*. If $n = 1$, then $\pi_s(R)$ has only one block, and $R[A,B]$ is called *trivially strong*.

NOTE 5. In the rest of the paper, and in general, a strong relation is always nontrivially strong unless otherwise specified.

THEOREM 8. *The family of strong relations is a proper subset of the family of relations*.

Proof. We give a simple example of a binary relation $R[A,B]$ which is not strong. Let

$$S_R[AB] = \{(a_1, b_1), (a_2, b_2), (a_1, b_2)\},$$

$$A = \{a_1, a_2\},$$

$$B = \{b_1, b_2\}.$$

It is obvious that there are only four different partitions of this relation in which $n>1$, and all four are not strong, because we do not have $\pi_I(A)$ and $\pi_I(B)$ for any of them.

DEFINITION 8. A partition

$$R[A,B]=\prod_{i=1}^{n} W_i[A_i,B_i] \qquad \text{where} \quad n\geqslant 1$$
$$=\pi_p(R)$$

of a binary relation $R[A,B]$ is a *prime partition* if

$$A=\prod_{i=1}^{n} A_i \qquad B=\prod_{i=1}^{n} B_i$$
$$=\pi_I(A), \qquad =\pi_I(B).$$

Here each block of $\pi_p(R)$ is a WCR, and $R[A,B]$ is called a *prime relation*. If $n=1$, then $R[A,B]$ is a WCR and is trivially prime.

NOTE 6. In the rest of this paper, and in general, a prime relation is always nontrivially prime unless otherwise specified.

THEOREM 9. *Any prime relation has one and only one prime partition.*

Proof. Let $R[A,B]$ be a prime relation. By definition a prime partition $\pi_{p_1}(R)$ exists. To show that $\pi_{p_1}(R)$ is unique, assume another prime partition $\pi_{p_2}(R)$ exists and

$$\pi_{p_1}(R)\neq\pi_{p_2}(R).$$

Let

$$\pi_{p_1}(R)=\prod_{i=1}^{n} W_{1i}[A_{1i},B_{1i}],$$

$$\pi_{p_2}(R)=\prod_{i=1}^{m} W_{2i}[A_{2i},B_{2i}],$$

and let

$$A=\prod_{i=1}^{n} A_{1i}=\prod_{i=1}^{m} A_{2i}$$

and

$$B=\prod_{i=1}^{n} B_{1i}=\prod_{i=1}^{m} B_{2i}.$$

$\therefore$
$$(\exists i)(\forall j)(W_{1i}\neq W_{2j})(1<i<n, 1<j<m).$$

But

$$S_{W_{1i}}\in S_R.$$

$\therefore$
$$(\exists j)(\exists k)(W_{1i}\cap W_{2j}\neq\varnothing \text{ and } W_{1i}\cap W_{2k}\neq\varnothing).$$

Let

$$W_{1i}=W_{1i}[A_{1i},B_{1i}],$$

$$W_{2j}=W_{2j}[A_{2j},B_{2j}],$$

$$W_{2k}=W_{2k}[A_{2k},B_{2k}].$$

$\therefore$
$$A_{1i}\cap A_{2j}\neq\varnothing,$$

$$A_{1i}\cap A_{2k}\neq\varnothing,$$

$$B_{1i}\cap B_{2j}\neq\varnothing,$$

$$B_{1i}\cap B_{2k}\neq\varnothing,$$

from Theorem 4.

Without loss of generality, we can assume that W_{2j} and W_{2k} are the only two WCRs in $\pi_{p_2}(R)$ which have common elements with W_{1i}. This is because the following argument can be extended to the more complicated case where more than two WCRs in $\pi_{p_2}(R)$ are assumed to have common elements with W_{1i}.

Let

$$a\in A_{1i},$$

$$b_1\in B_{1i}\cap B_{2j},$$

$$b_2\in B_{1i}\cap B_{2k}.$$

$\therefore$
$$\left.\begin{array}{l}(a,b_1)\in S_{W_{1i}},\\ (a,b_2)\in S_{W_{1i}}\end{array}\right\}\qquad (W_{1i} \text{ is a WCR}).$$

$\therefore$ (a,b_1) and (a,b_2) occur in W_{2j} and W_{2k}. We have three cases:

Case i: $(a,b_1)\in W_{2j}$ and $(a,b_2)\in W_{2k}$.

$$\therefore \qquad a\in A_{2j} \quad \text{and} \quad a\in A_{2k},$$

i.e., $\pi_{p_2}(R)$ is not a partition.

Case ii: $(a,b_1)\in W_{2j}$ and $(a,b_2)\in W_{2j}$. Then

$$b_1\in B_{2j}, \qquad b_2\in B_{2j}.$$

But

$$b_2\in B_{2k}$$

$$\therefore \qquad B_{2j} \text{ and } B_{2k} \text{ are not disjoint,}$$

i.e., $\pi_{p_2}(R)$ is not a partition.

Case iii: $(a,b_1)\in W_{2k}$ and $(a,b_2)\in W_{2k}$. Then

$$b_1\in B_{2k}, \qquad b_2\in B_{2k}.$$

But

$$b_1\in B_{2j}$$

$$\therefore \qquad B_{2j} \text{ and } B_{2k} \text{ are not disjoint,}$$

i.e., $\pi_{p_2}(R)$ is not a partition.

Hence every case leads to a contradiction.

$$\therefore \qquad \pi_{p_1}(R)=\pi_{p_2}(R)=\text{prime partition of } R[A,B].$$

THEOREM 10. *The family of prime relations is a proper subset of the family of strong relations.*

Proof. We give a simple example of a strong relation which is not prime. Let $R[A,B]$ be the strong relation. Then

$$S_R[AB]=\{(a_1,b_1),(a_1,b_2),(a_2,b_2),(a_3,b_3)\}.$$

$R[A,B]$ has only one partition with image partitions on A and B, and this is not a prime partition.

ALGORITHM 2. *To find the prime partition of a prime relation, $R[A,B]$.*

1. *Find set $B=\{b_1,b_2,\ldots,b_n\}$ where $b_i \neq b_j$ for $i \neq j$ and $1 \leqslant i,j \leqslant n$.*
2. *Find image sets $S_R[b_i]$ for $1 \leqslant i \leqslant n$.*
3. *Let $S_R[b_i]=A_i$, and find image sets $S_R[A_i]$ for $i \leqslant i \leqslant n$.*
4. *Let $S_R[A_i]=B_i$, and form WCRs $W_i[A_i,B_i]$ for $i \leqslant i \leqslant n$.*
5. *Then*

$$\pi_p(R) = \text{the prime partition of } R[A,B]$$

$$= \prod_{i=1}^{m} W_i[A_i,B_i] \text{ for } 1 \leqslant i \leqslant m,$$

where $m \leqslant n$, duplicate WCRs have been removed and the remaining WCRs have been renumbered from 1 to m.

Proof. To show that

(i) $R[A,B]=\bigcup_{i=1}^{m} W_i[A_i,B_i]$,
(ii) $A_i \cap A_j = \varnothing$ and $B_i \cap B_j = \varnothing$ for $i \leqslant i,j \leqslant m$ and $i \neq j$,
(iii) $W_i[A_i,B_i] \cap W_j[A_j,B_j] = \varnothing$ for $1 \leqslant i,j \leqslant m$ and $i \neq j$.

(i) Let $(a,b_i) \in R[A,B]$.

$\therefore$ $$a \in S_R[b_i],$$

i.e., a $\in A_i$.

$\therefore$ $$b_i \in S_R[A_i],$$

i.e., $b_i \in B_i$.

$\therefore$ $$(a,b_i) \in W_i[A_i,B_i]$$

$$\in \bigcup_{i=1}^{m} W_i[A_i,B_i].$$

$\therefore$ $$R[A,B] \subseteq \bigcup_{i=1}^{m} W_i[A_i,B_i].$$

Again, let

$$(a,b_i)\in \bigcup_{i=1}^{m} W_i[A_i,B_i]$$

$$\in W_i[A_i,B_i].$$

$\therefore$ $$b_i\in B_i$$

$$\in S_R[A_i]$$

and

$$a\in A_i$$

$$\in S_R[b_i],$$

i.e., $(a,b_i)\in R[A,B]$.

$\therefore$ $$\bigcup_{i=1}^{m} W_i[A_i,B_i]\subseteq R[A,B].$$

Hence $R[A,B]=\bigcup_{i=1}^{m} W_i[A_i,B_i]$.

(ii) Let $A_i\cap A_j\neq\varnothing$ for $i\neq j$ and $1\leqslant i,j\leqslant m$, and let

$$a\in A_i$$

$$\in A_j.$$

But $A_i=S_R[b_i]$ and $A_j=S_R[b_j]$.

$\therefore$ $$(a,b_i)\in W_i[A_i,B_i],$$

$$(a,b_j)\in W_j[A_j,B_j].$$

But

$$S_R[A_i]=B_i,$$

$$S_R[A_j]=B_j.$$

$\therefore$ $$S_R[a]=B_i=B_j \qquad (\because\ W_i \text{ and } W_j \text{ are WCRs}),$$

i.e., $S_R[A_i] = S_R[A_j]$. Again, if $B_i = B_j$,

$$S_R[B_i] = S_R[B_j],$$

i.e., $A_i = A_j$, $\because$ $W_i[A_i, B_i] = W_j[A_j, B_j]$ (from Theorem 5). But duplicate WCR's have been removed in step 5 of the algorithm. So $i = j$—a contradiction.

$$\therefore \qquad A_i \cap A_j = \varnothing \quad \text{for} \quad i \neq j \text{ and } i \leqslant i, j \leqslant m. \tag{33}$$

Again, let

$$B_i \cap B_j \neq \varnothing \quad \text{for} \quad i \neq j \text{ and } i \leqslant i, j \leqslant m.$$

Let

$$\begin{aligned} b_p &\in B_i \\ &\in B_j. \end{aligned}$$

Then

$$\left.\begin{aligned} S_R[b_p] &= A_i \\ &= A_j \end{aligned}\right\} \quad (\because \; W_i \text{ and } W_j \text{ are WCRs}).$$

$$\therefore \qquad A_i = A_j.$$

Hence $i = j$ [from (33)]. This is a contradiction. Hence,

$$B_i \cap B_j = \varnothing \quad \text{for} \quad i \neq j \text{ and } 1 \leqslant i, j \leqslant m. \tag{34}$$

(iii) This follows from (33) and Theorem 4.

Definition 9. An *elementary* WCR, $W[A, B]$, is one in which the second constituent B has a single element.

A *trivial* WCR, $W[A, B]$, is one in which both the constituents have a single element.

Definition 10. A *functional relation* $R[A, B]$ is a prime relation in which the prime partition has only elementary WCRs.

Theorem 11. *The family of functional relations is a proper subset of the family of prime relations.*

Proof. We give a simple example of a prime relation $R[A,B]$ which is not functional:

$$S_R[AB]=\{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_3)\}.$$

THEOREM 12. *In a functional relation, the canonical partition is equal to the prime partition.*

Proof. Let $R[A,B]$ be a functional relation, $\pi_c(R)$ be the canonical partition given by Algorithm 1, and $\pi_p(R)$ be the prime partition given by Algorithm 2. Assume $\pi_c(R)\neq\pi_p(R)$. Let

$$\pi_c(R)=\prod_{i=1}^{n} W_{ci}[A_{ci},B_{ci}]$$

and

$$\pi_p(R)=\prod_{i=1}^{m} W_{pi}[A_{pi},B_{pi}].$$

$$\therefore \qquad (\exists i)(\forall j)(W_{ci}\neq W_{pj})(i\leqslant i\leqslant n,\ 1\leqslant j\leqslant m).$$

But $W_{ci}[A_{ci},B_{ci}]$ is an elementary WCR (from Definitions 6 and 9), i.e.,

$$B_{ci}=\{b_i\},\qquad \text{where}\quad b_i\in B.$$

Then

$$\begin{aligned} S_R[b_i]&=A_{ci} &&\text{[from Algorithm 1]}\\ &=A_{pi} &&\text{[from Algorithm 2]},\\ S_R[A_{pi}]&=B_{pi} &&\text{[from Algorithm 2]}. \end{aligned}$$

From the previous two steps,

$$b_i\in B_{pi}.$$

But $R[A,B]$ is a functional relation and $W_{pi}[A_{pi},B_{pi}]$ is an elementary WCR;

$$\therefore \qquad B_{pi}=\{b_i\}=B_{ci},$$

$$\therefore \qquad W_{ci}=W_{pi} \qquad \text{[from Theorem 5]}.$$

This is a contradiction. Hence,

$$\pi_c(R) = \pi_p(R).$$

THEOREM 13. *The family of functional relations on sets A and B is in one to one correspondence with the family of nonpartial, onto functions from A to B.*

Proof. Let $R[A,B]$ be a functional relation. Define $f: A \to B$ as

$$f(a) = b \quad \text{if and only if} \quad (a,b) \in S_R[AB].$$

To show that

(i) $f: A \to B$ is functional,
(ii) $f: A \to B$ is nonpartial,
(iii) $f: A \to B$ is onto.

(i) Assume $f: A \to B$ is nonfunctional, i.e.,

$$f(a) = b_1 \text{ and } b_2, \qquad \text{where} \quad b_1 \neq b_2.$$

$$\therefore \qquad (a,b_1) \in S_R[AB] \quad \text{and} \quad (a,b_2) \in S_R[AB].$$

$R[A,B]$ is a functional relation;

$$\therefore \qquad R[A,B] = \prod_{i=1}^{n} W_i[A_i, B_i] \qquad \text{[by Algorithm 1]}.$$

Consider $W_1[A_1, B_1]$ and $W_2[A_2, B_2]$, where

$$B_1 = \{b_1\} \quad \text{and} \quad B_2 = \{b_2\}.$$

$$\therefore \qquad S_R[B_1] = A_1 \quad \text{and} \quad S_R[B_2] = A_2 \qquad \text{[from Algorithm 1]}.$$

$$\therefore \qquad a \in A \quad \text{and} \quad a \in A_2 \qquad (\because\ (a,b_1) \text{ and } (a,b_2) \in S_R[AB]),$$

i.e., $A_1 \cap A_2 \neq \varnothing$, i.e., $R[A,B]$ is not a functional relation. This is a contradiction.

(ii) From Definition 1,

$$(\forall a)(a \in A)(\exists b)(f(a) = b).$$

(iii) From Note 1(ii), it follows that $f: A \to B$ is onto.

Again, let $f:A\to B$ be a nonpartial, onto function. Define $R[A,B]$, a binary relation, as

$$(a,b)\in S_R[AB] \quad \text{if and only if} \quad f(a)=b.$$

To show

(i) $R[A,B]$ is non partial and onto,
(ii) $R[A,B]$ is a functional relation.

(i) This follows from the definition of $R[A,B]$ and the fact that $f:A\to B$ is nonpartial and onto.

(ii) We first prove that

$$R[A,B]=\bigcup_{i=1}^{n} W_i[A_i,B_i],$$

where

$$B_i=\{b_i\} \qquad \text{for} \quad 1\leqslant i\leqslant n$$

and

$$B=\{b_1,b_2,\ldots,b_n\}, \qquad \text{where} \quad b_i\neq b_j \quad \text{for } i\neq j,$$

and

$$A_i=S_R[B_i].$$

Let $(a,b_i)\in R[A,B]$. But $\{b_i\}=B_i$.

$$\therefore \qquad S_R[B_i]=A_i$$

and $a\in A_i$.

$$\therefore \qquad (a,b_i)\in W_i[A_i,B_i].$$

Again, let $(a,b_i)\in \bigcup_{i=1}^{n} W_i[A_i,B_i]$. Specifically, let

$$(a,b_i)\in W_i[A_i,B_i].$$

$$\therefore \qquad B_i=\{b_i\}$$

and $a \in A_i$. But $A_i = S_R[B_i]$.

$$\therefore \qquad (a,b_i) \in R[A,B].$$

$$\therefore \qquad R[A,B] = \bigcup_{i=1}^{n} W_i[A_i,B_i].$$

But $W_i \cap W_j = \varnothing$ for $i \neq j$ and $1 \leqslant i, j \leqslant n$. This follows from Theorem 4 together with

$$B_i = \{b_i\} \quad \text{and} \quad B_j = \{b_j\}$$

and

$$b_i \neq b_j \qquad \text{for} \quad i \neq j.$$

$$\therefore \qquad R[A,B] = \coprod_{i=1}^{n} W_i[A_i,B_i].$$

In the above, the r.h.s. would be a prime partition if

$$A_i \cap A_j = \varnothing \qquad \text{for} \quad i \neq j, \ 1 \leqslant i, j \leqslant n.$$

Assume

$$A_i \cap A_j \neq \varnothing.$$

Let

$$a \in A_i$$

$$\in A_j,$$

$$A_i = S_R[b_i],$$

$$A_j = S_R[b_j].$$

$$\therefore \qquad (a,b_i) \in R[A,B] \quad \text{and} \quad (a,b_j) \in R[A,B],$$

i.e.,

$$f(a) = b_i \quad \text{and} \quad f(a) = b_j,$$

i.e., $f:A \to B$ is not a function. This is a contradiction. Hence $R[A,B]$ is a prime relation and

$$R[A,B] = \prod_{i=1}^{n} W_i[A_i,B_i]$$

$$= \pi_p(R).$$

But $W_i[A_i,B_i]$ is an elementary WCR for all $1 \leqslant i \leqslant n$ by definition. Hence $R[A,B]$ is a functional relation.

DEFINITION 11. A bijectional relation $R[A,B]$ is a functional relation in which the prime partition has only trivial WCRs.

THEOREM 14. *The family of bijectional relations is a proper subset of the family of functional relations.*

Proof. We give an example for a functional relation $R[A,B]$ which is not a bijectional relation:

$$S_R[A\ B] = \{(a_1,b_1),(a_2,b_1),(a_3,b_2)\}.$$

THEOREM 15. *The family of bijectional relations on sets A and B is in one to one correspondence with the family of bijections from A to B.*

Proof. The proof is simple and on the same lines as that of Theorem 13.

NOTE. From this point, we use the following interchangeably:

(i) "functional relation" and "nonpartial onto function,"
(ii) "bijectional relation" and "nonpartial, onto bijection."

THEOREM 16. *Any binary relation lies somewhere in the following hierarchy:*

$$\mathfrak{F}_R \supset \mathfrak{F}_S \supset \mathfrak{F}_p \supset \mathfrak{F}_F \supset \mathfrak{F}_B,$$

where

$\mathfrak{F}_R$ = family of binary relations,

$\mathfrak{F}_S$ = family of strong relations,

$\mathfrak{F}_P$ = family of prime relations,

$\mathfrak{F}_F$ = family of functional relations,

$\mathfrak{F}_B$ = family of bijectional relations.

Proof. Obvious from Theorems 8, 10, 11 and 14.

PART III

The concepts of universal algebra are well known [7]. In this section, we apply some of these concepts to the family of prime relations. Most of the universal algebraic concepts used here are from the chapter on Algebraic Structures in [7].

It is well known that the concepts of morphisms, congruence relations and functions can be interrelated [7]. We show that the concepts of "morphic relations," congruence relations and prime relations can also be interrelated in a very similar way, thus generalizing the known interrelations.

We first define an "algebra of WCRs," an "image algebra" and a "morphic relation." We then show that morphic relations are a more general concept than morphisms (Theorem 17). We then show the interrelation between equivalence/congruence relations and the hierarchy of binary relations (Theorems 18 and 19). We then show the interrelation between quotient algebras and the algebra of WCRs (Theorem 20). Finally the morphic congruence theorem (Theorem 21) follows very simply.

DEFINITION 12. An *algebra* of WCRs, $\mathcal{A}_p$, over a prime relation P, is a pair (P, Ω) where Ω is a set of operations on the WCRs in the prime partition $\pi_p(P)$, i.e.,

$$P = \prod_{i=1}^{n} W_i = \pi_p(P),$$

$$\omega[W_1, W_2, \ldots, W_k] = W_j,$$

where $W_i \in \pi_p(P)$ for $1 \leqslant i \leqslant k$ and $W_j \in \pi_p(P)$,

$$\omega \in \Omega,$$

$$k = \text{"Arity" of } \omega,$$

$$(\pi_p, \Omega) = \mathcal{A}_p.$$

DEFINITION 13. An *image algebra* $\mathcal{I}_A(\mathcal{A}_p)$ is an algebra (A, Ω') such that:

(i) $P[A, B]$ is a prime relation.
(ii) $\mathcal{A}_p = (\pi_p, \Omega)$.
(iii) For every $\omega \in \Omega$ there exists a unique $\omega' \in \Omega'$.
(iv) Both ω and ω' are of the same arity.
(v) If $\omega[W_1, W_2, \ldots, W_k] = W_j$, then

$$\omega'[a_1, a_2, \ldots, a_k] = a_j,$$

where $a_i \in A_i$ for $1 \leqslant i \leqslant k$ and $a_j \in A_j$, with A_i, A_j the first constituents of W_i, W_j.

(vi) $(A, \Omega') = \mathcal{G}_A(\mathcal{Q}_p)$.

DEFINITION 14. A prime relation p is *morphic* if $\mathcal{Q}_p$, $\mathcal{G}_A(\mathcal{Q}_p)$ and $\mathcal{G}_B(\mathcal{Q}_p)$ exist.

THEOREM 17. *The family of onto morphisms is in one to one correspondence with the family of morphic, functional relations.*

Proof. We show that for every morphism there exists a morphic, functional relation and vice versa.

(i) Let $f: A \rightarrow B$ be a morphism where f is onto and nonpartial (follows easily from the definition of morphisms), i.e., there exist two algebras (A, Ω_A) and (B, Ω_B) such that for every ω_{A_1} in Ω_A there is a unique ω_{B_1} in Ω_B, ω_{A_1} and ω_{B_1} are of the same arity, and

$$f(\omega_{A_1}(a_1, a_2, \ldots, a_k)) = \omega_{B_1}(f(a_1), f(a_2), \ldots, f(a_k)) = b_j. \tag{35}$$

Now f is a functional relation (from Theorem 13 and Note 7). Let the prime partition of f be

$$\pi_p(f) = \prod_{i=1}^{n} W_i[A_i, B_i]. \tag{36}$$

Define $\mathcal{Q}_p = (\pi_p, \omega)$, where for every pair $\omega_{A_1}, \omega_{B_1}$ there is a unique ω of the same arity and

$$\omega(W_1, W_2, \ldots, W_k) = W_j, \tag{37}$$

where

$$a_i \in A_i \text{ in } W_i[A_i, B_i] \quad \text{for} \quad 1 \leqslant i \leqslant k$$

and

$$b_j \in B_j \text{ in } W_j[A_j, B_j].$$

To show that $\mathcal{G}_A(\mathcal{Q}_p) = (A, \Omega_A)$ and $\mathcal{G}_B(\mathcal{Q}_p) = (B, \Omega_B)$. From Definition 13, conditions (i) to (iv) are met by the construction of $\mathcal{Q}_p$. In (36),

$$W_i[A_i, B_i] \text{ for } 1 \leqslant i \leqslant n \text{ are elementary WCRs}$$

($\because f$ is functional), i.e., $B_i = \{b_i\} =$ a single element set. Again, in (37),

$$a_i \in A_i \text{ and } f(a_i) = b_i \quad \text{for} \quad 1 \leqslant i \leqslant k.$$

Hence from (35),

$$\omega_{B_1}(f(a_1),f(a_2),\ldots,f(a_k))=\omega_{B_1}(b_1,b_2,\ldots,b_k)=b_j,$$

where

$$b_1\in B_1,\quad b_2\in B_2,\quad \text{and so on.}$$

Hence $\mathcal{G}_B(\mathcal{Q}_p)=(B,\Omega_B)$. Again, from (35),

$$f(\omega_{A_1}(a_1,a_2,\ldots,a_k))=\omega_{B_1}(b_1,b_2,\ldots,b_k)=b_j,$$

where

$$a_1\in A_1,\quad a_2\in A_2,\quad \text{and so on.}$$

This is true even if a_1, $a_2,\ldots$ are replaced by other elements in $A_1, A_2,\ldots,$ i.e.,

$$f(\omega_{A_1}(a_{11},a_{21},\ldots,a_{k1}))=\omega_{B_1}(f(a_{11}),f(a_{21}),\ldots,f(a_{k1})),$$

where $a_{11}\in A_1$, $a_{21}\in A_2$, and so on. But

$$f(a_{11})=b_1,\quad f(a_{21})=b_2,\quad \text{and so on.}$$

To prove this, let

$$\begin{aligned} f(a_{11}) &\neq b_1 \\ &= b_i \qquad \text{where}\quad i\neq 1, \end{aligned}$$

i.e.,

$$a_{11}\in A_i \text{ in } W_i[A_i,B_i].$$

But

$$a_{11}\in A_1 \text{ in } W_1[A_1,B_1],$$

i.e., $A_i\cap A_1\neq\varnothing$ (violates definition of prime partition).

$$\therefore\qquad f(a_{11})=b_1,\quad f(a_{21})=b_2,\quad \text{and so on.}$$

$$\therefore\qquad \begin{aligned} f(\omega_{A_1}(a_{11},a_{21},\ldots,a_{k1})) &= \omega_{B_1}(b_1,b_2,\ldots,b_k) \\ &= b_j. \end{aligned}$$

Hence from (37) and the above expression, it follows that

$$\mathcal{G}_A(\mathcal{Q}_p) = (A, \Omega_A)$$

(ii) Let $f[A,B]$ be a morphic, functional relation;

$$\therefore \qquad \pi_p(f) = \prod_{i=1}^{n} W_i[A_i, B_i]$$

and

$$\mathcal{Q}_p = (\pi_p, \Omega),$$

where $\omega(W_1, W_2, \ldots, W_k) = W_j$ for $\omega \in \Omega$. From Theorem 13 and Note 7,

f is a non partial, onto function, $f: A \to B$.

By definition of a morphic, functional relation,

$\mathcal{G}_A(\mathcal{Q}_p)$ and $\mathcal{G}_B(\mathcal{Q}_p)$ exist.

It is a simple matter to show that

$$\mathcal{G}_A(\mathcal{Q}_p) = (A, \Omega_A),$$

$$\mathcal{G}_B(\mathcal{Q}_p) = (B, \Omega_B),$$

and (35) is true in this case, making f a morphism.

THEOREM 18. *If $R[A,B]$ is prime and each WCR in the prime partition has equal constituents, then R is an equivalent relation, and vice versa.*

Proof. We show that $R[A,B]$ is reflexive, symmetric and transitive. Let

$$\pi_p(R) = \prod_{i=1}^{n} W_i[A_i, B_i],$$

where $A_i = B_i$ for $1 \leqslant i \leqslant n$. Consider $a, b, c \in A$. Let $a \in A_i$. Then

$$a \in B_i \qquad (\because A_i = B_i)$$

and

$$aRa \qquad (\because\ W_i[A_i,B_i] \text{ is a WCR}).$$

Hence $R[A,B]$ is reflexive.

Again, let

$$a \in A_i \quad \text{and} \quad aRb.$$

Then

$$b \in B_i \qquad (\because\ W_i[A_i,B_i] \text{ is a WCR in } \pi_p(R)).$$

$$\therefore \qquad b \in A_i \text{ and } a \in B_i \qquad (\because\ A_i = B_i)$$

$$\therefore \qquad bRa \qquad (\because\ W_i[A_i,B_i] \text{ is a WCR}).$$

Hence $R[A,B]$ is symmetric.

Again, let

$$aRb, \quad bRc \quad \text{and} \quad a \in A_i.$$

$$\therefore \qquad b \in B_i \quad \text{and} \quad b \in A_i$$

($\because\ W_i[A_i,B_i]$ is a WCR and $A_i = B_i$).

$$\begin{array}{ll} \therefore & c \in B_i \\ \therefore & aRc \end{array} \Bigg\} \qquad (\because\ W_i[A_i,B_i] \text{ is a WCR})$$

Hence $R[A,B]$ is transitive.

To prove the converse, we can assume $R[A,B]$ to be reflexive, symmetric and transitive and $A = B$. A partition of A (and hence B) exists because R is an equivalence relation.

Consider one class $[a]$ of elements related to a where $a \in A$ and $a \in B$. Let $R_1[[a],[a]] \subseteq R[A,B]$ be a relation such that

$$R_1[[a],[a]] = \{(a,b) : (a \in [a]), (b \in [a]), (aRb)\}.$$

Then R_1 is a WCR whose constituents are equal (trivially true).

$$\therefore \qquad R[A,B] = R_1 \cup R_2 \cup \cdots \cup R_n,$$

where n is the number of elements in A (or B). Again,

$$R_i \cap R_j = \varnothing$$

($\because$ each is a WCR and their constitutents are disjoint) for $i \neq j$ and $1 \leqslant i, j \leqslant n$.

$$\therefore \qquad R[A,B] = \prod_{i=1}^{n} R_i = \pi_p(R).$$

Here $\pi_p(R)$ is prime and each WCR has equal constituents.

THEOREM 19. *If $R[A,B]$ is equivalent and morphic, then it is a congruence relation, and vice versa.*

Proof. Let

$$R[A,B] = \prod_{i=1}^{n} W_i[A_i, B_i] = \pi_p(R)$$

and $\mathcal{Q}_p$, $\mathcal{G}_A(\mathcal{Q}_p)$ and $\mathcal{G}_B(\mathcal{Q}_p)$ exist. This follows from Theorem 18 and definition of a morphic relation. Further,

$$A = B,$$

$$A_i = B_i \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

and

$$\mathcal{G}_A(\mathcal{Q}_p) = \mathcal{G}_B(\mathcal{Q}_p)$$

from Theorem 18 and the definition of an image algebra.

NOTE 8. [Implicit here is the definition that two algebras are equal, i.e.

$$\mathcal{G}_A(\mathcal{Q}_p) = \mathcal{G}_B(\mathcal{Q}_p)$$

if $A = B$ and the operations are identical for both.]

Let

$$\mathcal{Q}_p = (\pi_p, \bar{\Omega}), \qquad \mathcal{G}_A(\mathcal{Q}_p) = \mathcal{G}_B(\mathcal{Q}_p) = (A, \Omega_A) = (B, \Omega_B).$$

$$\therefore \qquad \omega_1(W_1, W_2, \ldots, W_k) = W_j \qquad \text{for} \quad \omega_1 \in \Omega$$

and

$$\omega_{A_1}(a_1,a_2,\ldots,a_k)=a_j$$

for $\omega_{A_1}\in\Omega_A$, $a_1\in A_1$, $a_2\in A_2$ and so on,

$$\omega_{B_1}(b_1,b_2,\ldots,b_k)=b_j$$

for $\omega_{B_1}\in\Omega_B$, $b_1\in B_1$, $b_2\in B_2$ and so on. But

$$A_1=B_1, \qquad A_2=B_2,$$

and so on ($\because$ R is an equivalence relation). Also

$$a_1Rb_1, \qquad a_2Rb_2,$$

and so on ($\because$ $W_1, W_2,\ldots$ are WCRs). Similarly,

$$a_j\in A_j, \quad b_j\in B_j, \quad A_j=B_j \text{ and } a_jRb_j.$$

Lastly

$$\Omega_A=\Omega_B \quad\text{and}\quad \omega_{A_1}=\omega_{B_1}.$$

Hence,

$$a_iRb_i \qquad \text{for} \quad 1\leqslant i\leqslant k$$

$$\Rightarrow \quad \omega_A(a_1,a_2,\ldots,a_k)=\omega_A(b_1,b_2,\ldots,b_k).$$

This is the substitution property, and hence $R[A,B]$ is a congruence relation.

Conversely, let R be a congruence relation on an algebra (A,Ω_A), i.e.,

$$a_iRb_i \qquad \text{for} \quad 1\leqslant i\leqslant k$$

$$\Rightarrow \quad \omega_A(a_1,a_2,\ldots,a_k)=\omega_A(b_1,b_2,\ldots,b_k) \qquad \text{for} \quad \omega_A\in\Omega_A.$$

From Theorem 18, $R[A,A]$ is prime and each WCR of the prime partition has equal constituents. Let

$$R[A,A]=\prod_{i=1}^{n} W_i[A_i,A_i]=\pi_p(R).$$

To show that $R[A,A]$ is morphic, we have to show that $\mathcal{Q}_p$ and $\mathcal{G}_A(\mathcal{Q}_p)$ exist. Let

$$\mathcal{Q}_p = (\pi_p, \Omega)$$

be such that

$$\omega(W_1, W_2, \ldots, W_k) = W_j \quad \text{if and only if} \quad \omega_A(a_1, a_2, \ldots, a_k) = a_j,$$

where

$$a_1 \in A_1, \quad a_2 \in A_2, \ldots, \quad a_j \in A_j, \quad \omega \in \Omega \text{ and } \omega_A \in \Omega_A.$$

$\mathcal{Q}_p$ is unique because of the following: if

$$\omega_A(b_1, b_2, \ldots, b_k) = b_j,$$

where $a_1 \neq b_1$, $a_2 \neq b_2, \ldots, a_j \neq b_j$ and

$$a_1 R b_1, \quad a_2 R b_2, \ldots, \quad a_j R b_j,$$

then

$$b_1 \in A_1, \quad b_2 \in A_2, \ldots, \quad b_j \in A_j$$

($\because$ $\pi_p(R)$ is a prime partition). Hence again we get

$$\omega(W_1, W_2, \ldots, W_k) = W_j.$$

Also

$$\mathcal{G}_A(\mathcal{Q}_p) = (A, \Omega_A) \qquad \text{(trivially true).}$$

Hence $R[A,A]$ is morphic.

DEFINITION 15. Let (A, Ω_A) be an algebra and R a congruence relation; then the *quotient algebra* of R, i.e. $(A/R, \Omega)$, is

$$\omega([a_1], [a_2], \ldots, [a_k]) = [\omega_A(a_1, a_2, \ldots, a_k)],$$

where

$$\omega_A \in \Omega_A \quad \text{and} \quad \omega \in \Omega.$$

THEOREM 20. *The quotient algebra of a congruence relation $R[A,A]$ is isomorphic to the algebra of WCRs, $\mathcal{Q}_p(R)$.*

Proof. All this theorem states is that the quotient set A/R can be replaced by the set of WCRs in the prime partition of $R[A,A]$. The proof is simple and is left as an exercise for the reader.

THEOREM 21 (Morphic-congruence theorem).

(i) *On two sets A and B, if a morphic relation $R[A,B]$ exists, then two congruence relations, C_A on A and C_B on B, always exist such that the quotient algebras are isomorphic to the algebra of WCRs.*

(ii) *On two sets A and B, if two congruence relations C_A on A and C_B on B exist such that the quotient algebras are isomorphic, then a morphic relation $R[A,B]$ always exists.*

Proof. (i) Let $R[A,B]$ be a morphic relation, i.e.,

$$R[A,B]=\prod_{i=1}^{n} W_i[A_i,B_i]=\pi_p(R),$$

and

$$\mathcal{Q}_p=(\pi_p,\Omega),$$

$$\mathcal{G}_A(\mathcal{Q}_p)=(A,\Omega_A),$$

$$\mathcal{G}_B(\mathcal{Q}_p)=(B,\Omega_B)$$

all exist. Also,

$$A=\prod_{i=1}^{n} A_i$$

and

$$B=\prod_{i=1}^{n} B_i$$

(from the definition of a prime partition). Define C_A as follows:

$$(a\,C_A\,b) \quad \text{if and only if}$$

$$(\exists i)(1\leqslant i\leqslant n)\ (a\in A_i) \text{ and } (b\in A_i).$$

Similarly C_B is given by

$$(a\,C_B\,b) \quad \text{if and only if}$$

$$(\exists i)(1 \leqslant i \leqslant n)\ (a \in B_i) \text{ and } (b \in B_i).$$

$$\therefore \qquad A/C_A = \{A_1, A_2, \ldots, A_n\}$$

and

$$B/C_B = \{B_1, B_2, \ldots, B_n\}.$$

Let the quotient algebras be

$$(A/C_A, \Omega_1) \text{ and } (B/C_B, \Omega_2)$$

such that

$$\omega_1(A_1, A_2, \ldots, A_k) = \left[\omega_{A_1}(a_1, a_2, \ldots, a_k)\right],$$

where $\omega_1 \in \Omega_1$, $\omega_{A_1} \in \Omega_A$ and $a_1 \in A_1$, $a_2 \in A_2$, and so on, and

$$\omega_2(B_1, B_2, \ldots, B_k) = \left[\omega_{B_2}(b_1, b_2, \ldots, b_k)\right],$$

where $\omega_2 \in \Omega_2$, $\omega_{B_2} \in \Omega_B$ and $b_1 \in B_1$, $b_2 \in B_2$, and so on. The proof that $\mathcal{C}_p$, $(A/C_A, \Omega_1)$ and $(B/C_A, \Omega_2)$ are isomorphic follows easily from the above definitions and is left as an exercise.

(ii) Let C_A and C_B be two congruence relations, and let $(A/C_A, \Omega_1)$ and $(B/C_B, \Omega_2)$ be two quotient algebras which are isomorphic, i.e.,

$$f(\omega_1([a_1], [a_2], \ldots, [a_k])) = \omega_2(f[a_1], f[a_2], \ldots, f[a_k]),$$

where

$$f : A/C_A \leftrightarrow B/C_B \qquad \text{(i.e. a bijection)},$$

$$\omega_1 \in \Omega_1,$$

and

$$\omega_2 \in \Omega_2.$$

Then define $R[A,B]$ as

$$R[A,B]=\prod_{i=1}^{n} W_i[[a_i], \qquad f[a_i]]=\pi_p(R),$$

and

$$\mathcal{Q}_p=(\pi_p,\Omega),$$

where

$$\begin{aligned}\omega(W_1,W_2,\ldots,W_k)&=W_j[\omega_1([a_1],[a_2],\ldots,[a_k]),\omega_2(f[a_1],f[a_2],\ldots,f[a_k])]\\&=W_j[[a_j],f[a_j]]\end{aligned}$$

for $\omega\in\Omega$. The proof that $R[A,B]$ is morphic follows from the above definitions and the definition of an image algebra, and is left as an exercise. Here the image algebras are in fact the original algebras based on which the quotient algebras were defined.

CONCLUSION

The theory developed in this paper has been applied to the dependency structures in a relational data base which were originally studied by Codd [2, 3]. When one tries to express functional dependencies and multivalued dependencies in terms of WCRs, the more general contextual dependencies follow in a natural way. (It is to be noted that contextual dependencies are the same as Nicolas's mutual dependencies. However the approaches taken by Nicolas and us differ considerably. Our contextual dependencies only confirm Nicolas's conclusions). In Part III we have studied the family of prime relations from a universal algebraic point of view. In particular we have defined an algebra of WCRs. The types of operations on WCRs have been left out and have been studied elsewhere [8]. One can conceive of several operations to manipulate WCRs. For example, in a canonical partition, the WCRs are all elementary and we can think of the union, intersection, difference and so on of WCRs belonging to two binary relations. Other operations can be thought of if one considers the two canonical partitions in a binary relation—namely, one in which all WCRs have a single element in the first constituent, and the other in which they have a single element in the second constituent. Now any intersection of two WCRs from these two partitions will always give a single "tuple" of the binary relation or null. This could be thought of as the basis of conversion from a canonical partition to the trivial partition in which every "tuple" is a WCR.

Lastly we note two interesting points. Firstly, in [14] the concept of a DBTG set has been introduced. A DBTG set is nothing but a partition of elementary WCRs. Secondly, in [15] Fagin has introduced the concept of a boolean dependency. We note that any boolean dependency describes some particular arrangement of elementary WCRs.

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