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Application of moment and Fourier descriptors to the accurate estimation of elliptical-shape parameters

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Abstract

Safaee-Rad, R., K.C. Smith, B. Behnhabib and I. Tchoukanov, Application of moment and Fourier descriptors to the accurate estimation of elliptical-shape parameters, Pattern Recognition Letters 13 (1992) 497-508.

Accurate estimation of the parameters of an elliptical shape is required in various machine-vision and computer-vision problems. In our previous work, we have addressed this problem by proposing to optimize a weighted minimum-squares-error (MSE) function. As a continuation of this work, we have studied other techniques for elliptical-parameter estimation, ones applying elliptical-Fourier descriptors, moments of area, and moments of perimeter. A study, to be reported here, was carried out to evaluate the comparative performance of the above-mentioned three techniques and the one based on the weighted MSE function. The limitation and degree of accuracy of each technique was determined. It was found that different elliptical-parameter-estimation techniques must be applied depending on acceptable computational cost, number of parameters to be estimated, the required degree of accuracy, and the specific conditions under which the estimation must be performed.

1. Introduction

Accurate estimation of the five basic parameters of an elliptical shape (namely, the center coordinates, the major and minor radii, and the orientation) arises in various machine-vision-related problems: (1) in pattern recognition and scene analysis [1]; (2) in machine-vision metrology [2]; and (3) in 3D-location estimation in both its direct and inverse forms [3].

Basically, there have been two methods used for dealing with the elliptical-shape-parameter-estimation problem. The first is based on the use of optimization techniques in specific contexts. From a purely mathematical point of view, the problem of fitting a conic or a conic section to a set of data has been addressed in various papers [4]. The same problem has been addressed in the applied literature as well: in dentistry, for the estimation of dental arch from [5], in biology, for automatic chromosome analysis [6], in manufacturing, for quality estimation of mechanical parts [7], in object recognition, for detecting cylindrical parts and their orientation [8]; and in pattern recognition and scene analysis, for the reconstruction problem [9].

The second method used for dealing with the same problem is based on the Hough transformation in various forms: the Hough transformation itself [10], modified Hough transformation [11], decomposed five-dimensional Hough transformation [12], etc.

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In our previous work [13,14], we have proposed a new technique for this purpose: an optimal fit of an elliptical curve to an elliptical shape using a weighted minimum-squares error (MSE) function. The weighting factor was derived using a new geometrical interpretation of the least-squares error function.

As a continuation of this work, we have studied other techniques for elliptical-parameter estimation [15]. These techniques are different from the two general methods previously used for this purpose by various researchers. They are based on moments (both area and perimeter) and elliptical-Fourier descriptors. That is, we are proposing to use these shape-specific features to estimate the five basic parameters of an elliptical shape. Herein, the complete report on the proposed methods is presented.

In Section 2 of this paper, first, we address the mathematics involved in the estimation of Fourier expansion series coefficients of a closed contour, based on which the parameters of the approximating ellipse can be estimated (Section 2.1). Subsequently, we address the estimation of the approximating ellipse of a closed contour based on moments. In this context, first, we present the moments of *area*, and then propose an alternative method for the estimation of area moments based on Gauss' theorem (Section 2.2). Section 2.3 addresses the mathematics involved in the moments of *perimeter*. In Section 2.4, we briefly discuss the weighted minimum-squares error function that we have used earlier for elliptical-parameters estimation [13,14]. In Section 3, first, we propose an objective and independent measure for 'goodness' of the results of various methods. Subsequently, we present experimental data obtained by applying the above methods to two different cases. Conclusions are provided in the final section.

2. Parameter estimation of elliptical shapes

2.1. Elliptical-Fourer descriptors

A continuous, closed contour in two dimensions can be represented parametrically as a function of time t, $V(t)$, assuming that the contour is traced at a constant speed. The projections of the vector function $V(t)$ on the x and y axes are $x(t)$ and $y(t)$. These functions are periodic with period T, where T is the total time required to trace the whole contour at the constant speed. Furthermore, they can be represented in Fourier trigonometric series as follows:

$$
\begin{aligned}
x(t) &= A_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}, \\
y(t) &= C_0 + \sum_{n=1}^{\infty} c_n \cos\frac{2n\pi t}{T} + d_n \sin\frac{2n\pi}{T}
\end{aligned}
\tag{1}
$$

where

$$
A_0 = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t, \qquad a_n = \frac{2}{T}\int_0^T x(t)\cos\frac{2n\pi t}{T}\,\mathrm{d}t, \qquad b_n = \frac{2}{T}\int_0^T x(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t. \tag{2}
$$

For C_0, c_n and d_n, similar expressions can be derived.

Different levels of approximation to a closed contour can be obtained by using a different number of harmonics. In general, the truncated Fourier approximation of a closed boundary based on the first N harmonics can be written as:

$$
x_N(t) = A_0 + \sum_{n=1}^{N} X_n(t), \qquad y_N(t) = C_0 + \sum_{n=1}^{N} Y_n(t) \tag{3}
$$

where

$$
X_n(t) = a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}, \qquad Y_n(t) = c_n \cos\frac{2n\pi t}{T} + d_n \cos\frac{2n\pi t}{T}. \tag{4}
$$

Kuhl and Giardina [16] have shown that the locus of each vector of constant frequency is elliptical; that is, if

$$S_n(t) = \begin{pmatrix} X_n(t) \\ Y_n(t) \end{pmatrix} \tag{5}$$

is the vector of harmonic frequency n ($n>0$), then by removing the variable t in the equations (4), the following equation is obtained:

$$(c_n^2 + d_n^2)X_n^2 + (a_n^2 + b_n^2)Y_n^2 - 2(a_n c_n + b_n d_n)X_n Y_n - (a_n d_n - b_n c_n)^2 = 0 \tag{6}$$

which clearly represents an elliptical locus (for a given n), since

$$4(a_n c_n + b_n d_n)^2 - (c_n^2 + d_n^2)(a_n^2 + b_n^2) < 0. \tag{7}$$

Furthermore, they showed that for any piecewise-linear representation of a contour—irrespective of the incremental changes in ΔX and ΔY—(for example, in a Freeman chain-encoded contour), the following expressions for the Fourier coefficients can be derived [16]:

$$\begin{aligned} a_n &= \frac{T}{2n^2\pi^2} \sum_{p=1}^{K} \frac{\Delta x_p}{\Delta t_p} \left[\cos\frac{2n\pi t_p}{T} - \cos\frac{2n\pi t_{p-1}}{T} \right], \\ b_n &= \frac{T}{2n^2\pi^2} \sum_{p=1}^{K} \frac{\Delta x_p}{\Delta t_p} \left[\sin\frac{2n\pi t_p}{T} - \sin\frac{2n\pi t_{p-1}}{T} \right] \end{aligned} \tag{8}$$

where K is the total number of linear segments comprising the boundary (linear links), Δx_p and Δy_p are the lengths of the *projections* of the linear link p on the x and y axes, and $\Delta t_p = t_p - t_{p-1}$ is the required time to trace the link p at a constant speed. Similar expressions can be derived for the coefficients c_n and d_n. The DC-components in the Fourier series for piecewise linear contours can be expressed as follows:

$$\begin{aligned} A_0 &= \frac{1}{T} \sum_{p=1}^{K} \frac{\Delta x_p}{2\Delta t_p}(t_p^2 - t_{p-1}^2) + \zeta_p(t_p - t_{p-1}), \\ C_0 &= \frac{1}{T} \sum_{p=1}^{K} \frac{\Delta y_p}{2\Delta t_p}(t_p^2 - t_{p-1}^2) + \delta_p(t_p - t_{p-1}) \end{aligned} \tag{9}$$

where

$$\zeta_p = \sum_{j=1}^{p-1} \Delta x_j - \frac{\Delta x_p}{\Delta t_p} \sum_{j=1}^{p-1} \Delta t_j, \qquad \delta_p = \sum_{j=1}^{p-1} \Delta y_j - \frac{\Delta y_p}{\Delta t_p} \sum_{j=1}^{p-1} \Delta t_j \quad \text{and} \quad \zeta_1 = \delta_1 = 0.$$

Based on (3) and (6), it can be concluded that the Fourier approximation to the original contour can be viewed as the addition, in proper phase, of rotating phasors, each of which has an elliptical locus [16].

If the boundary itself is an ellipse, as in the case under study, it is expected that the Fourier approximation, based on the first harmonic ($n = 1$), would lead to a good approximation of the elliptical boundary and of its five basic parameters. Furthermore, it is expected that as the resolution of boundary points increases, the accuracy of the estimated elliptical shape parameters will increase as well (for example, using subpixel edge-point data rather than simple pixel edge-point data).

Thus taking the first harmonic, that is, using the Fourier coefficients a_1, b_1, c_1, and d_1, the five elliptical-shape parameters are estimated using the following relations:

$$X_0 = A_0, \qquad Y_0 = C_0,$$

$$\theta = \arctan\left[\frac{(c-a) + \sqrt{(c-a)^2 + b^2}}{b} \right],$$

$$A^2 = \left[\frac{2f}{b^2 - 4ac} \right] [(c+a) + \sqrt{(c-a)^2 + b^2}],$$

$$B^2 = \left[\frac{2f}{b^2-4ac}\right][(c+a)-\sqrt{(c-a)^2+b^2}], \tag{10}$$

where

$$a = c_1^2+d_1^2, \qquad b=-2(a_1c_1+b_1d_1), \qquad c=a_1^2+b_1^2, \qquad f=-(a_1d_1-b_1c_1)^2.$$

These coefficients of equation (6) (a, b, c, and f) are referred to as elliptical-Fourier descriptors.

2.2. *Area-moment descriptors*

The $(p+q)$th order moments of a two-dimensional density function $\varrho(x,y)$ are defined in terms of Riemann integrals as [17],

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q \varrho(x,y)\,dx\,dy, \quad p,q=0,1,2,\ldots . \tag{11}$$

If the density function $\varrho(x,y)$ is a binary-valued picture S, then m_{pq} would be simplified (assuming $\varrho(x,y)=1$ for points of S) as

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q \,dx\,dy \quad \text{for } all\ (x,y)\in S. \tag{12}$$

The above definition of moments is based on *area*. Thus, if the binary image is run-length-coded, the following formulae can be used for the estimation of moments of area [18]:

$$\begin{aligned}
m_{00} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} 1 = \sum_{y=j}^{k}(n_y-m_y),\\
m_{10} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} x = \sum_{y=j}^{k}\tfrac{1}{2}[(n_y-m_y)(m_y+n_y-1)],\\
m_{01} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} y = \sum_{y=j}^{k}[y(n_y-m_y)],\\
m_{20} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} x^2 = \sum_{y=j}^{k}\tfrac{1}{12}[3(n_y-m_y)(m_y+n_y-1)^2+(n_y-m_y)^3-(n_y-m_y)],\\
m_{11} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} xy = \sum_{y=j}^{k}\tfrac{1}{2}(y(n_y-m_y)(m_y+n_y-1)],\\
m_{02} &= \sum_{y=j}^{k}\sum_{x=m_y}^{n_y-1} y^2 = \sum_{y=j}^{k}[y^2(n_y-m_y)].
\end{aligned} \tag{13}$$

Based on the above set of moments, Agin [18] gives the following formulae for the parameters of the approximating ellipse of a 2D shape:

$$X_0 = m_{10}/m_{00}, \qquad Y_0 = m_{01}/m_{00},$$

$$A = \sqrt{\frac{a+b+e}{2f}}, \qquad B=\sqrt{\frac{a+b-e}{2f}}, \qquad \theta = \frac{1}{2}\arctan\frac{2c}{a-b} \tag{14}$$

where

$$a = \frac{4}{\pi}\left(m_{20}-\frac{m_{10}^2}{m_{00}}\right), \qquad b=\frac{4}{\pi}\left(m_{02}-\frac{m_{01}^2}{m_{00}}\right), \qquad c=\frac{4}{\pi}\left(m_{11}-\frac{m_{11}}{m_{00}}\right),$$

$$e = \sqrt{(a-b)^2+\frac{4}{\pi}c^2}, \qquad f=(ab-c^2)^{1/4}.$$

Equations (13) can be used only if the edge-point data are digitized and run-length coded. As a result, it cannot be applied to subpixel edge-point data. Here, an alternative method is proposed for area-moment estimation based on Gauss' theorem:

$$\iint_{(R)} (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y = \int_{(C)} P\,\mathrm{d}x + Q\,\mathrm{d}y, \tag{15}$$

where (R) represents the area bounded by the closed contour (C). Let

$$Q = \frac{x^{p+1}y^q}{p+1}, \qquad P = \frac{x^p y^{q+1}}{q+1}, \tag{16}$$

then the moment m_{pq} (equation (1[illegible]) can be expressed as follows, using Gauss' theorem:

$$m_{pq} = \iint_{(R)} x^p y^q\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{p+1}\int_{(C)} x^{p+1}y^q\,\mathrm{d}y = -\frac{1}{q+1}\int_{(C)} x^p y^{q+1}\,\mathrm{d}x. \tag{17}$$

Thus, the first six moments of area (the zeroth, first, and second moments) can be expressed as:

$$\begin{aligned}
m_{00} &= \int_{(C)} x\,\mathrm{d}y = -\int_{(C)} y\,\mathrm{d}x,\\
m_{10} &= \frac{1}{2}\int_{(C)} x^2\,\mathrm{d}y = -\int_{(C)} xy\,\mathrm{d}x,\\
m_{01} &= \int_{(C)} xy\,\mathrm{d}y = -\frac{1}{2}\int_{(C)} y^2\,\mathrm{d}x,\\
m_{20} &= \frac{1}{3}\int_{(C)} x^3\,\mathrm{d}y = -\int_{(C)} x^2 y\,\mathrm{d}x,\\
m_{11} &= \frac{1}{2}\int_{(C)} x^2 y\,\mathrm{d}y = -\frac{1}{2}\int_{(C)} xy^2\,\mathrm{d}x,\\
m_{02} &= \int_{(C)} xy^2\,\mathrm{d}y = -\frac{1}{3}\int_{(C)} y^3\,\mathrm{d}x.
\end{aligned} \tag{18}$$

Using the mean-value theorem for integrals, the moments (18) can be expressed for any piecewise-linear representation of a contour as follows:

$$\begin{aligned}
m_{00} &= \sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)(y_{i+1}-y_i),\\
m_{10} &= \frac{1}{2}\sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)^2 (y_{i+1}-y_i),\\
m_{01} &= \sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)\left(\frac{y_{i+1}+y_i}{2}\right)(y_{i+1}-y_i),\\
m_{20} &= \frac{1}{3}\sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)^3 (y_{i+1}-y_i),\\
m_{11} &= \frac{1}{2}\sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)^2 \left(\frac{y_{i+1}+y_i}{2}\right)(y_{i+1}-y_i),\\
m_{02} &= \sum_{i=1}^{n} \left(\frac{x_{i+1}+x_i}{2}\right)\left(\frac{y_{i+1}+y_i}{2}\right)^2 (y_{i+1}-y_i)
\end{aligned} \tag{19}$$

where n is the total number of linear segments (links) of a closed contour. Note that if the boundary of a shape is represented by Freeman chain code, the term $(y_{i+1}-y_i)$ can be either 0 or ± 1, a property which significantly reduces the computational cost.

From a geometrical point of view, the zeroth moment m_{00} represents the area of a closed contour (as is defined in (12)), and the first moments, m_{10} and m_{01}, are related to the centroid coordinates of a shape as are expressed in (14). To estimate the other three parameters of the approximating ellipse, the second moments must be used as follows. The orientation parameter can be estimated using the following formula:

$$\theta = \frac{1}{2}\arctan\frac{2m_{11}}{m_{02}-m_{20}}. \tag{20}$$

The two values for θ are 90 degrees apart. To estimate the major and minor radii, first the central second moments $(\bar{m}_{11}, \bar{m}_{20}, \bar{m}_{02})$ are estimated using the centroid coordinates, and then they are normalized with respect to area. Subsequently, the covariance matrix of second moments is defined as,

$$C = \begin{pmatrix} \bar{m}_{20}/m_{00} & \bar{m}_{11}/m_{00} \\ \bar{m}_{11}/m_{00} & \bar{m}_{02}/m_{00} \end{pmatrix}. \tag{21}$$

The two eigenvalues of matrix C, λ_1 and λ_2, are used to estimate the major and minor radii as follows:

$$A = 2\sqrt{\lambda_1}, \qquad B = 2\sqrt{\lambda_2}. \tag{22}$$

From a geometrical point of view, λ_1 and λ_2 represent the normalized (with respect to area) values of maximum and minimum moments of inertia of a planar shape. Formulae (22) can be proven by analytically solving the second-moment integrals (12) for an elliptical shape. To show this by example, the simpler case is considered when the second central moments are estimated with respect to the major and minor axes of an elliptical shape, in which case the following relations are obtained:

$$\bar{m}_{11} = 0, \quad \bar{m}_{20} = \frac{\pi}{4}A^3B, \quad \bar{m}_{02} = \frac{\pi}{4}AB^3, \quad \lambda_1 = \frac{A^2}{4}, \quad \lambda_2 = \frac{B^2}{4} \tag{23}$$

which shows the validity of (22).

2.3. *Perimeter-moment descriptors*

In Section 2.2, the moments of *area* were addressed. As an alternative, moments of a closed contour may be defined based on the *perimeter* of a shape.

To define perimeter moments, mathematically, the definition of moments of area (12) can be used. But this time, just a strip of area which approximates a line segment (i.e., the boundary of a shape—L) is considered. Thus, if a constant cross-section D (Figure 1) is assumed, the following can be written:

$$\mathrm{d}A = D\,\mathrm{d}l. \tag{24}$$

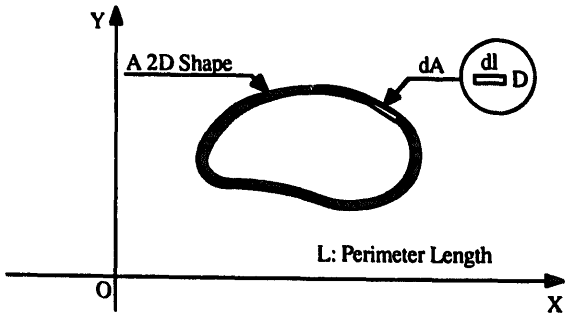

Figure 1. Schematic representation of a differential element of area dA.

Then (12) is simplified as

$$j_{pq} = D \int_{(C)} x^p y^q \, \mathrm{d}l \quad \text{for } all\ (x, y) \in L. \tag{25}$$

Now, if j_{pq} is *normalized* with respect to the perimeter of a shape (L) and denoted by J_{pq}, the following is obtained:

$$J_{pq} = D \int_{(C)} x^p y^q \, \mathrm{d}l \Big/ D \int_{(C)} \mathrm{d}l = \frac{1}{L} \int_{(C)} x^p y^q \, \mathrm{d}l. \tag{26}$$

Using the mean-value theorem for integrals, the normalized perimeter moment (26) for any piecewise-linear representation of a contour can be expressed as follows:

$$J_{pq} = \frac{1}{L} \sum_{i=1}^{n} \left(\frac{x_{i+1} + x_i}{2} \right)^p \left(\frac{y_{i+1} + y_i}{2} \right)^q l_i \tag{27}$$

where

$$l_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}, \qquad L = \sum_{i=1}^{n} l_i$$

and n is the total number of linear segments of a closed contour.

The zeroth moment (j_{00}) clearly represents the perimeter length of a shape. The first normalized moments J_{10} and J_{01} represent the centroid coordinates of a shape. To determine the orientation of the approximating ellipse of a closed contour, the following formula can be applied:

$$\theta = \frac{1}{2} \arctan \frac{2 j_{11}}{j_{02} - j_{20}}. \tag{28}$$

For estimation of the major and minor radii of the approximating ellipse, unfortunately, due to the nature of the line integral (25), it is not possible to derive an exact analytical expression that defines the two radii in terms of the second-order moments J_{20} and J_{02}. To show this, the following integral is considered:

$$j_{02} = \int_{(C)} y^2 \, \mathrm{d}l \qquad \text{where } \mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}. \tag{29}$$

If an ellipse at its standard position is considered, the following can be written:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \tag{30}$$

based on which, the above integral is simplified (assuming $A > B$) as:

$$j_{02} = \frac{B^2}{A^3} \sqrt{A^2 - B^2} \int_{(C)} \sqrt{(A^2 - x^2)\left(\frac{A^4}{A^2 - B^2} - x^2 \right)} \, \mathrm{d}x. \tag{31}$$

Unfortunately, there is *no exact* solution for the above integral and it must be *approximated*. As a result, it is not possible to establish a definite analytical relationship between the radii of the approximating ellipse and the second normalized perimeter moments J_{11}, J_{20}, and J_{02}.

2.4. *Weighted minimum-squares-error function*

Let

$$Q(X, Y) = aX^2 + bXY + cY^2 + dX + eY + f = 0 \tag{32}$$

be the general equation of an ellipse. Let (X_i, Y_i), $i = 1, \ldots, N$, be a set of points to which an elliptical shape

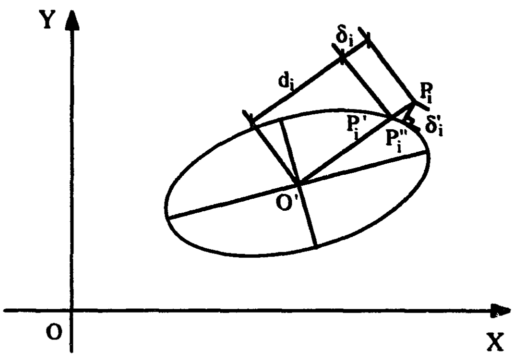

Figure 2. Distances d_i, δ_i, and δ_i' of a data point to an ellipse.

is to be fitted. Then, the minimum-squares-error (MSE) function is defined as

$$J_0 = \sum_{i=1}^{N} [Q(X_i, Y_i)]^2. \tag{33}$$

Thus, the objective is to determine a parameter vector $W^T = (a, b, c, d, e, f)$.

It has been shown that the contribution of the data points to the above MSE function is not uniform [19]. To minimize the influence of the nonuniformity of the contribution of the data points to the error function, the following weighting factor has been derived based on a new geometrical interpretation of the error function J_0 [13, 14]:

$$w_i = \left(\frac{d_i}{A}\right)\left[\left(1 + \frac{\delta_i}{2A}\right) \Big/ \left(1 + \frac{\delta_i}{2d_i}\right)\right] \tag{34}$$

where A is the major radius of the ellipse, $d_i = P_i'O'$, and $\delta_i = P_iP_i'$ (Figure 2). Based on the weighting factor defined above, the weighted MSE function is defined as follows:

$$J_1 = \sum_{i=1}^{N} [w_i Q(X_i, Y_i)]^2_{f=1}. \tag{35}$$

Using J_0 for an initial 'optimal' ellipse, we can estimate δ_i and d_i for each data point. Thereafter, the minimization of the error function J_1 would proceed by taking the first derivatives with respect to the five unknowns (a, b, c, d, e), to yield a set of five linear equations with five unknowns, the solution of which is the vector $W^T = (a, b, c, d, e)$. The five parameters of the final optimal ellipse can then be estimated using the following formulae,

$$X_0 = \frac{2cd - be}{b^2 - 4ac}, \qquad Y_0 = \frac{2ae - bd}{b^2 - 4ac},$$

$$\theta_A = \arctan\left[\frac{(c-a) + \sqrt{(c-a)^2 + b^2}}{b}\right],$$

$$A^2 = \left[\frac{2(1 - F_S)}{b^2 - 4ac}\right][(c+a) + \sqrt{(c-a)^2 + b^2}],$$

$$B^2 = \left[\frac{2(1 - F_S)}{b^2 - 4ac}\right][(c+a) - \sqrt{(c-a)^2 + b^2}]$$

where

$$F_S = \frac{bde - ae^2 - cd^2}{b^2 - 4ac}. \tag{36}$$

Figure 3. A simulated ellipse. (b) An imaged ellipse (thresholded).

3. Experimental data analysis

In order to carry out a comparative study of the four different methods proposed in Section 2, an objective and independent measure should be applied to determine the level of 'goodness' of fit. In this section, such an objective measure is defined as 'the sum of normal distances of all the data points to the approximating or optimal ellipse'. To express this in mathematical form, let δ_i' be the normal distance of a data point i to the approximating or optimal ellipse ($P_i P_i''$ in Figure 2). Then, the 'goodness' measure is defined as:

$$G = \sum_{i=1}^{N} \delta_i'. \tag{37}$$

The method that yields a smaller value of G, is a more accurate elliptical-parameter-approximation one. For more details on this measure refer to [13, 14].

The methods presented in Section 2 were applied to two different cases: a simulated ellipse and a distorted-imaged ellipse. In the first case, a 'perfect' ellipse is generated using digitizer-board graphic commands, the boundary of which is digitized according to a methametical procedure that minimizes the digitization error of a continuous boundary of an ellipse (Figure 3(a)) [20]. In the second case, an 'imperfect' ellipse image was used to simulate possible external distortions such as the thresholding effect on a grey-level image (Figure 3(b)). The shape of this ellipse is further distorted by its passage through the image-acquisition system (that is, the camera and the digitizer board).

The experimental results for the first case are summarized in Table 1. Based on G values (equation (37)), the weighted MSE function yields the most accurate estimates of the ellipse's parameters. Thus, it is used as a *reference*. We can conclude the following from the table: For a 'perfect' ellipse (1) All methods yield the same estimates for center coordinates and orientation angle of the approximating ellipse; the differences manifest themselves in the estimates of the major and minor radii. (2) Due to digitization error (which results in sharp vertex angles—90 and 135 degrees), the first harmonic of the Fourier expansion leads to

Table 1
Experimental results for a simulated ellipse ($N = 268$)

Method	X_0	Y_0	A	B	θ	Area	G
Generated ellipse	150.000	329.000	60.000	30.000	90.000	5654.867	64.175
Elliptical Fourier desc.	150.000	329.000	54.915	31.963	90.000	—	—
Area moments[a]	150.000	329.000	60.497	30.291	90.000	5757.000	124.578
Area moments[b]	150.000	329.000	60.067	29.794	90.000	5622.000	62.835
Perimeter moments	150.000	329.000	—	—	90.000	—	—
J1	150.000	329.000	60.072	29.796	90.000	—	62.783

N: total number of boundary points. Length unit: pixel. Angle unit: degrees.
[a] Based on run-length code. [b] Based on Gauss' theorem.

a significant error, in the estimates of A and B. (3) The results of the two algorithms based on area moments are not the same. The formulae based on run-length code yield estimates that are approximately 0.5 pixel in error, which is significant. As a result the G value for this method is significantly higher than for the method based on the J_1 error function. (4) Applying Gauss' theorem clearly results in a more accurate estimation of A and B and yields a G-value very close to the reference value. Furthermore, it is computationally cheaper than the formulae based on run-length code, and it can be applied to sub-pixel edge-data points as well. (5) Although the method based on perimeter moments does not provide estimates for A and B, it accurately estimates the other three parameters (as the method based on elliptical Fourier descriptors does).

The experimental results for the distorted-imaged ellipse are summarized in Table 2. In this case, area moments based on the Gauss' theorem yield the minimum G value (thus it is used as a reference), though the weighted MSE function yields a G value which is nearly the same. For the 'imperfect' imaged ellipse as well, the same trends are observed: (1) All methods result in relatively accurate estimates of X_0, Y_0, and θ. (2) The method based on elliptical-Fourier descriptors yields a significant error in the estimates of A and B. This is expected due to quantization errors. (3) Estimation of A and B using area moments that are based on run-length code are approximately 0.5 pixel in error. (4) Applying Gauss' theorem to area-moment estimation leads to a more accurate parameter estimation of an ellipse and to less computational cost.

Based on both sets of results, we can present the following *general* conclusions:

(a) If a segment of an ellipse's boundary is available, the only accurate method that can be used is an elliptical-curve fitting technique—that is, the method based on the weighted MSE function.

(b) If the whole boundary of an ellipse (whether sampled or not) is available, and an accurate estimation of all parameters of an ellipse is required, both the weighted MSE function and area moments (based on Gauss' theorem) can be used.

(c) If the centroid coordinates and orientation of an ellipse are required, all the methods addressed in this paper provide relatively accurate estimates; though the two methods highlighted in (b) above are more accurate.

(d) Application of Gauss' theorem to area-moment estimation clearly improves the area-moment-based technique in the following sense: (1) It increases the accuracy of area-moment estimation; (2) it can be applied to sub-pixel edge-point data; and (3) it is computationally cheaper. The fundamental reason due to which this method leads to a more accurate estimation lies in the interpretation of a pixel as a point rather than as a unit area: That is, in the case of area-moment estimation based on run-length code, a pixel is assumed to be a unit of area; while, in the case of application of Gauss' theorem to area-moment estimation, a pixel represents a point without area. This difference is clearly seen in the results of area estimation of the simulated ellipse: The run-length-code based method yields 5757 squared pixel units, while the application of Gauss' theorem results in 5622 squared pixel units, while the actual area is 5654.87 squared pixel units. Clearly the application of Gauss' theorem leads to a better estimation. The above contradiction between a pixel as a point and a pixel as a unit area is a classical problem in computer vision [21].

Table 2
Experimental results for a simulated ellipse ($N=255$)

Method	X_0	Y_0	A	B	θ	Area	G
Imaged ellipse	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Elliptical Fourier desc.	98.255	415.790	52.921	31.998	30.998	—	—
Area moments[a]	98.384	415.711	58.238	30.333	30.594	5546.000	156.592
Area moments[b]	98.379	415.710	57.744	29.884	30.547	5417.500	114.386
Perimeter moments	98.255	415.790	—	—	30.246	—	—
J1	98.311	415.698	57.683	29.952	30.405	—	114.568

N: total number of boundary points. Length unit: Pixel. Angle unit: degrees.
[a] Based on run-length code. [b] Based on Gauss' theorem.

4. Conclusions

In this paper, the problem of accurate parameter estimation of elliptical shapes was addressed. Three methods were presented. These methods are based on elliptical-Fourier descriptors, area moments, and for perimeter moments of 2D elliptical shapes. The mathematics of these methods has been presented and, for area moments, a set of new formulae based on Gauss' theorem given. To implement a comparative performance study, a method previously developed (the weighted MSE function) was included and an objective and independent measure of 'goodness' of fit was used. The performance of these methods were compared for two different cases, those of a 'perfect' simulated ellipse and an 'imperfect' imaged ellipse. It was shown, based on analytical derivations and experimental results, that different elliptical-parameter-estimation techniques must be applied depending on acceptable computational cost, number of parameters to be estimated, the required degree of accuracy, and the specific conditions under which the estimation must be performed.

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